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ON THE SUP AND INF INVARIANCE OF SOME SEPARATION AXIOMS

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1. Introduction.

We consider here some of the most familiar separation axioms, namely T_4, $T_{3^1/_2}$, T_3, T_2, T_1, T_0, N (normality), CR (complete regularity), R (regularity), and R_0, the terminology for the T_ν 's being that of [3] or [7]. $T_{31/2}$-space means Tychonoff space. An R_0*-space* (also called *symmetric space*) is a topological space $\langle X, \mathfrak{T}\rangle$ satisfying the axiom

$$(R_0) \quad x, y \in X; \quad x \in cl_{\mathfrak{T}}\{y\} \Longrightarrow y \in cl_{\mathfrak{T}}\{x\}$$

(cf. [1] p. 889; [7], p. 49, Problem 9). It is well known that the implications

$$(1) \qquad \begin{array}{ccccccccccc} T_4 & \Longrightarrow & T_{31/2} & \Longrightarrow & T_3 & \Longrightarrow & T_2 & \Longrightarrow & T_1 & \Longrightarrow & T_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & & & \Downarrow & & \\ N & & CR & \Longrightarrow & R & & \Longrightarrow & & R_0 & & \end{array}$$

hold ([7], p. 61, Theorem 4. 3. 1). A property P which can be attributed to a topological space (or to a topology) is called *sup invariant* (*inf invariant*) if, for any family $(\mathfrak{T}_j)$ of topologies on a fixed non-empty set X, $\sup\limits_{j\in J} \mathfrak{T}_j = \bigvee\limits_{j\in J} \mathfrak{T}_j$ $(\inf\limits_{j\in J} \mathfrak{T}_j = \bigcap\limits_{j\in J} \mathfrak{T}_j)$ has property P whenever each $\mathfrak{T}_j$ has property P. A topology having property P is called a P topology.

The purpose of this note is to complete the statements about invariance or non-invariance of the separation axioms occurring in (1).

2. Sup invariance.

On the basis of topological embeddability of $\langle X, \bigvee\limits_{j\in J}\mathfrak{T}_j\rangle$ into the topological product of the $\langle X, \mathfrak{T}_j\rangle$'s, N. Levine ([4], Corollary 8) established the sup invariance of most of the separation axioms considered here. His method easily applies also to T_0 and R_0. (For a different procedure concerning R and CR cf. [6]). A remarkable exception from sup invariance is normality ([4], Example 17. 5); however, the counterexample given there does not serve for T_4.

Theorem 1. *T_4 is not sup invariant. More precisely, the sup of two T_4 topologies need not be normal.*

Proof. Let $\mathfrak{R}$ denote the usual topology and $\mathfrak{R}^+$ the right half-open interval topology ([7], p 24, Example 3) on the set $\mathbf{R}$ of real numbers. For simplicity we denote the product topology of two topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$ by $\mathfrak{T}_1 \times \mathfrak{T}_2$. Since $\mathfrak{R} \subset \mathfrak{R}^+$, it follows that.

$$(2) \qquad (\mathfrak{R} \times \mathfrak{R}^+) \vee (\mathfrak{R}^+ \times \mathfrak{R}) \subset \mathfrak{R}^+ \times \mathfrak{R}^+ \text{-}$$

($\mathfrak{R}^+ \times \mathfrak{R}^+$ is called Sorgenfrey's half-open rectangle topology). By virtue of $E_1 := (a-1, b) \times [c, d) \in \mathfrak{R} \times \mathfrak{R}^+$, $E_2 := [a, b) \times (c-1, d) \in \mathfrak{R}^+ \times \mathfrak{R}$, $[a, b) \times [c, d) = E_1 \cap E_2 \in (\mathfrak{R} \times \mathfrak{R}^+) \vee (\mathfrak{R}^+ \times \mathfrak{R})$ for any real numbers a, b, c, d such that $a < b$ and $c < d$, and of (2) we obtain

$$(3) \qquad (\mathfrak{R} \times \mathfrak{R}^+) \vee (\mathfrak{R}^+ \times \mathfrak{R}) = \mathfrak{R}^+ \times \mathfrak{R}^+.$$

Now, for $I := [0, 1]$, we define the subspace topologies

$$(4) \qquad \mathfrak{T}_1 := (\mathfrak{R} \times \mathfrak{R}^+) \,|\, (I \times I), \quad \mathfrak{T}_2 := (\mathfrak{R}^+ \times \mathfrak{R}) \,|\, (I \times I).$$

It is known that $\mathfrak{T}_1 = \mathfrak{R} | I \times \mathfrak{R}^+ | I$ ([2], p. 99, 1. 2. (3)). As a closed subspace of the T_3 Lindelöf space $\langle \mathbf{R}, \mathfrak{R}^+ \rangle$ ([7], p. 78, Example 2), $\langle I, \mathfrak{R}^+ | I \rangle$ is T_3 Lindelöf, therefore T_2 and paracompact by Morita's theorem ([2], p. 174, Theorem 6. 5). This and the fact that $\mathfrak{R} | I$ is T_2 and compact imply that $\mathfrak{T}_1$ and — analogously —$\mathfrak{T}_2$ are T_4 topologies ([5], p. 180, Theorem V. 7).

On the other hand, by (3), (4), and the easily verified formula $\mathfrak{T}_1 \vee \mathfrak{T}_2 = [(\mathfrak{R} \times \mathfrak{R}^+) \vee (\mathfrak{R}^+ \times \mathfrak{R})] \,|\, (I \times I)$, we get $\mathfrak{T}_1 \vee \mathfrak{T}_2 = (\mathfrak{R}^+ \times \mathfrak{R}^+) \,|\, (I \times I)$. The points of $I \times I$ having both coordinates rational form a $\mathfrak{T}_1 \vee \mathfrak{T}_2$ — dense subset of $I \times I$, so $\mathfrak{T}_1 \vee \mathfrak{T}_2$ is separable. $A := \{(x, 1-x) : x \in I\}$ is a discrete closed subspace of $\langle I \times I, \mathfrak{T}_1 \vee \mathfrak{T}_2 \rangle$. By an argument of F. B. Jones ([2], p. 144, Example 3), $\mathfrak{T}_1 \vee \mathfrak{T}_2$ is not normal.

3. Inf invariance.

Here the situation is quite different from that of section 2.

Theorem 2. *Among the axioms in (1), T_1 is the only one which is inf invariant. More precisely, if P is an axiom occurring in (1) and being distinct from T_1, then the inf of two P topologies need not be a P topology.*

Proof. i) T_1 is inf invariant since, for every non-empty set X, the cofinite topology $\{G \subset X;\ X \setminus G \text{ finite}\} \cup \{\varnothing\}$ on X is included in any T_1 topology on X.

ii) Let X be an infinite set and p a fixed element of X. We define

$$(5) \qquad \mathfrak{T}_p := \{G \subset X;\quad X \setminus G \text{ finite and/or } p \notin G\}.$$

It is not hard to see that $\mathfrak{T}_p$ is a topology on X and that $\{q\} \in \mathfrak{T}_p$ for every q in $X \setminus \{p\}$. (This topology was used in [4], Example 17.1, for other purposes). The $\mathfrak{T}_p$-closed sets F are characterized by

$$(6) \qquad F \subset X, \quad F \text{ finite and/or } p \in F.$$

Let F_1, F_2 be two disjoint $\mathfrak{T}_p$-closed sets. Case 1: $p \in F_1 \cup F_2$, say $p \in F_1$. Then $p \notin F_2$ and, by virtue of (5), F_2 is $\mathfrak{T}_p$-open. Hence F_1 and F_2 can be separated by the open sets F_2 and $X\setminus F_2$. Case 2: $p\notin F_1\cup F_2$, i. e., $p\notin F_1$ and $p\notin F_2$. By (5), F_1 and F_2 are $\mathfrak{T}_p$-open and can be separated by F_1 and F_2. This shows that $\mathfrak{T}_p$ is normal. By (6), $\mathfrak{T}_p$ is also T_1, thus T_4. (By the way, $\mathfrak{T}_p$, as the one-point compactification of the discrete topology on $X\setminus\{p\}$, is compact and T_2, hence T_4; cf. [7], p. 139, Theorem 8.1.2, and p. 83, Theorem 5.4.7). As a consequence of (1), $\mathfrak{T}_p$ satisfies all the separation axioms occurring in (1).

Now let p, q be two distinct elements of X. Then $\{p\}$ and $\{q\}$ are disjoint, $\mathfrak{T}_p$-closed, and $\mathfrak{T}_q$-closed, therefore $\mathfrak{T}_p\cap\mathfrak{T}_q$-closed. Let G_1, $G_2\in\mathfrak{T}_p\cap\mathfrak{T}_q$ such that $p\in G_1$, $q\in G_2$. (5) and $G_1\in\mathfrak{T}_p$, $G_2\in\mathfrak{T}_q$ imply that $X\setminus G_1$ and $X\setminus G_2$ are finite. It follows that $X\setminus(G_1\cap G_2)$ is finite, i. e., $\neq X$, in other words that $G_1\cap G_2\neq\varnothing$. So $\{p\}$ and $\{q\}$ cannot be separated by $\mathfrak{T}_p\cap\mathfrak{T}_q$-open sets which means that $\mathfrak{T}_p\cap\mathfrak{T}_q$ is neither normal nor regular nor T_2. A fortiori, by (1), $\mathfrak{T}_p\cap\mathfrak{T}_q$ does not satisfy T_4, $T_{3 1/2}$, T_3, and CR. (For a different argument concerning T_2 cf. [7], p. 92, Problem 106).

iii) Of course, the counterexample of part ii) cannot serve for the separation axioms "beyond" T_1, namely for T_0 and R_0. But the T_0 case is settled by $X=\{a, b\}$, $a\neq b$, $\mathfrak{T}_1=\{\varnothing, \{a\}, X\}$, $\mathfrak{T}_2=\{\varnothing, \{b\}, X\}$, and the R_0 case by $X=\mathbf{R}$, $\mathfrak{T}_1$ the usual topology $\mathfrak{R}$ on $\mathbf{R}$, $\mathfrak{T}_2=\{\varnothing, (0,1), \mathbf{R}\setminus(0,1), \mathbf{R}\}$.

REFERENCES

[1] A. S. Davis, *Indexed systems of neighborhoods for general topological spaces*, Amer. Math. Monthly **68** (1961) 886—893.

[2] J. Dugundji, *Topology*, Allyn and Bacon, Boston 1967.

[3] J. L. Kelley, *General topology*, Van Nostrand, Princeton 1955.

[4] N. Levine, *On families of topologies for a set*, Amer. Math. Monthly **73** (1966) 358—361.

[5] J. Nagata, *Modern general topology*, North-Holland Publ. Comp., Amsterdam 1968.

[6] M. J. Norris, *A note on regular and completely regular topological spaces*, Proc. Amer. Math. Soc. **1** (1950) 754—755.

[7] A. Wilansky, *Topology for analysis*, Ginn, Waltham, Mass. 1970.

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