PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 22 (36), 1977, pp. 139—143

SOME EXAMPLES FOR NONLINEAR SUPERPOSITION

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(Received February 12, 1977)

0. In papers [1], [2], [3], [4] connecting functions for some partial differential equations are obtained.

Let u, v be solutions of some partial differential equation $N(U)=0$. Any function F depending on u and v, which is also a solution of $N(U)=0$ will be called a connecting function for the equation $N(U)=0$.

In this paper we propose a method for obtaining connecting functions for some second-order partial differential equations. These examples contain some results from [1], [2], [3], [4].

1. Start with the linear equation

$$\sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial V}{\partial x_i} + CV = 0, \tag{1.1}$$

where A_{ij}, B_i, C are functions of $x_1, \ldots, x_n$.

Let $g: R \to R$ be a twice differentiable function and let T be an operator defined on a certain set of functions which will be specified in examples. Furthermore suppose that g^{-1} and T^{-1} exist.

Let F satisfy the following equation

$$g(TF) = C_1 g(Tu) + C_2 g(Tv) \tag{1.2}$$

where C_1 and C_2 are constants. From (1.2) it follows

$$F = T^{-1}(g^{-1}(C_1 g(Tu) + C_2 g(Tv))). \tag{1.3}$$

Putting $V = g(TU)$, equation (1.1) becomes

$$\sum_{i,j=1}^{n} A_{ij} \left(g'(TU) \frac{\partial^2 TU}{\partial x_i \partial x_j} + g''(TU) \frac{\partial TU}{\partial x_i} \frac{\partial TU}{\partial x_j} \right) + \sum_{i=1}^{n} B_i g'(TU) \frac{\partial TU}{\partial x_i} + Cg(TU) = 0. \tag{1.4}$$

Let S be a linear operator defined on a set of functions, such that there exists S^{-1}. Then we have

$$(1.5)\qquad S\left(\sum_{i,j=1}^{n} A_{ij}\left(g'(TU)\frac{\partial^2 TU}{\partial x_i\,\partial x_j}+g''(TU)\frac{\partial TU}{\partial x_i}\frac{\partial TU}{\partial x_j}\right)+\sum_{i=1}^{n} B_i g'(TU)\frac{\partial TU}{\partial x_i}+Cg(TU)\right)=0.$$

The following statements are valid:

U is a solution of equation (1.5) if and only if U is a solution of equation (1.4);

If u, v are solutions of equations (1.4) and (1.5) then F, given by (1.3) is also a solution of (1.4) and (1.5), i.e. F is a connecting function for these equations.

Equation (1.5), in general, is not a second-order partial differential equation. If T and S are given, we determine, if possible, g, A_{ij}, B_i, C so that (1.5) becomes a second-order partial differential equation.

2. Examples

1° Let $TU=U$ and $SU=U$. Then, equation (1.5) becomes

$$\sum_{i,j=1}^{n} A_{ij}\left(g'(U)\frac{\partial^2 U}{\partial x_i\,\partial x_j}+g''(U)\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_j}\right)+\sum_{i=1}^{n} B_i g'(U)\frac{\partial U}{\partial x_i}+Cg(U)=0$$

and a connecting function is given by

$$F=g^{-1}\left(C_1 g(u)+C_2 g(v)\right).$$

This result is obtained by J. D. Kečkić in [1] and [2], and is some special cases, by S. A. Levin in [3].

2° Let $TU=\int_{x_n^0}^{x_n} U(x_1,\ldots,x_n)\,dx_n+\varphi(x_1,\ldots,x_{n-1})$ and let $SU=\dfrac{\partial U}{\partial x_n}$.

Then, we have

$$\frac{\partial}{\partial x_n}\left\{\sum_{i,j=1}^{n-1} A_{ij}\left[g''\cdot\left(\int_{x_n^0}^{x_n}\frac{\partial U}{\partial x_i}dx_n+\frac{\partial\varphi}{\partial x_i}\right)\cdot\left(\int_{x_n^0}^{x_n}\frac{\partial U}{\partial x_j}dx_n+\frac{\partial\varphi}{\partial x_j}\right)\right.\right.$$

$$\left.+g'\cdot\left(\int_{x_n^0}^{x_n}\frac{\partial^2 U}{\partial x_i\partial x_j}dx_n+\frac{\partial^2\varphi}{\partial x_i\,\partial x_j}\right)\right]$$

$$+\sum_{i=1}^{n-1}(A_{in}+A_{ni})\,g''\cdot\left(\int_{x_n^0}^{x_n}\frac{\partial U}{\partial x_i}dx_n+\frac{\partial\varphi}{\partial x_i}\right)\cdot U+g'\cdot\frac{\partial U}{\partial x_i}$$

$$\left.+A_{nn}\left(g''\,U^2+g'\frac{\partial U}{\partial x_n}\right)+\sum_{i=1}^{n-1}B_i g'\cdot\left(\int_{x_n^0}^{x_n}\frac{\partial U}{\partial x_i}dx_n+\frac{\partial\varphi}{\partial x_i}\right)+B_n g'\cdot U+Cg\right\}=0.$$

It follows upon differentation in x_n, that the above equation becomes a second-order partial differential equation if

$$g(t)=C\exp(\alpha t)\qquad (C,\ \alpha \text{ constants});$$

$$A_{ii}=0;\ \frac{\partial B_i}{\partial x_n}=0\qquad (i=1,\dots,n-1);$$

$$A_{ij}+A_{ji}=0\qquad (i,j=1,\dots,n;\qquad i\neq j).$$

Then we have

$$\alpha A_{nn}\frac{\partial^2 U}{\partial x_n^2}+2\alpha^2 A_{nn}U\frac{\partial U}{\partial x_n}+\alpha\left(\frac{\partial A_{nn}}{\partial x_n}+B_n\right)\frac{\partial U}{\partial x_n}+\alpha^2\frac{\partial A_{nn}}{\partial x_n}U^2+\alpha\sum_{i=1}^{n-1}B_i\frac{\partial U}{\partial x_i}+\alpha\frac{\partial B_n}{\partial x_n}U+\frac{\partial C}{\partial x_n}=0, \tag{2.1}$$

and φ satisfies the equation

$$\alpha A_{nn}\left(\frac{\partial U}{\partial x_n}+\alpha U^2\right)+\alpha B_n U+C+\alpha\sum_{i=1}^{n-1}B_i\left(\int_{x_n^0}^{x_n}\frac{\partial U}{\partial x_i}dx_n+\frac{\partial\varphi}{\partial x_i}\right)=0.$$

Since the function φ does not depend on x_n, for $x_n=x_n^0$ the above equation reduces to

$$\alpha\sum_{i=1}^{n-1}B_i\frac{\partial\varphi}{\partial x_i}+\left(\alpha A_{nn}\left(\frac{\partial U}{\partial x_n}+\alpha U^2\right)+\alpha B_n U+C\right)\Bigg|_{x_n=x_n^0}=0. \tag{2.2}$$

The function φ is not arbitrary, but has to be a solution of the above first-order partial differential equation.

Hence, we conclude: If u and v are solutions of equation (2.1), if φ_1 and φ_2 are solutions of equation (2.2) for $U=u$ and $U=v$, respectively, then

$$F=\frac{\partial}{\partial x_n}\left(\log\left(C_1\exp\left(\int_{x_n^0}^{x_n}u\,dx_n+\varphi_1\right)+C_2\exp\left(\int_{x_n^0}^{x_n}v\,dx_n+\varphi_2\right)\right)\right),$$

is also a solution of equation (2.1).

This result was proved in [2] by J. D. Kečkić by a different method. Also some special cases of the equation (2.1) and nonlinear superpositions for these equations are given in papers [3] and [4].

3° Let $TU=\dfrac{\partial U}{\partial x_n}$ and $SU=\displaystyle\int_{x_n^0}^{x_n}U\,dx_n$. Then we have

$$\int_{x_n^0}^{x_n}\left\{\sum_{i,j=1}^{n}A_{ij}\left(g'\left(\frac{\partial U}{\partial x_n}\right)\frac{\partial^3 U}{\partial x_i\,\partial x_j\,\partial x_n}+g''\left(\frac{\partial U}{\partial x_n}\right)\frac{\partial^2 U}{\partial x_i\,\partial x_n}\cdot\frac{\partial^2 U}{\partial x_j\,\partial x_n}\right)+\sum_{i=1}^{n}B_i g\left(\frac{\partial U}{\partial x_n}\right)\frac{\partial^2 U}{\partial x_i\,\partial x_n}+Cg\left(\frac{\partial U}{\partial x_n}\right)\right\}dx_n=0.$$

If $A_{ij}+A_{ji}=0,\ B_i=\frac{\partial}{\partial x_n}(A_{in}+A_{ni})\qquad (i,\ j=1,\dots,n-1),$

$$B_n=\frac{\partial A_{nn}}{\partial x_n}+D(x_1,\dots,x_n),\ C=\frac{\partial D}{\partial x_n},$$

then the above equation becomes

$$A_{nn}g'\left(\frac{\partial U}{\partial x_n}\right)\frac{\partial^2 U}{\partial x_n^2}+\sum_{i=1}^{n-1}(A_{in}+A_{ni})g'\left(\frac{\partial U}{\partial x_n}\right)\frac{\partial^2 U}{\partial x_i\,\partial x_n}+Dg\left(\frac{\partial U}{\partial x_n}\right)=0, \tag{2.3}$$

and we get the result: If u and v are solutions of equation (2.3) then

$$F=\int_{x_n^0}^{x_n} g^{-1}\left(C_1 g\left(\frac{\partial u}{\partial x_n}\right)+C_2 g\left(\frac{\partial v}{\partial x_n}\right)\right)dx_n+\varphi(x_1,\dots,x_{n-1})$$

is also a solution of this equation (φ is an arbitrary function).

4[0] Let $TU=\frac{\partial U}{\partial x_n}+U$ and $SU=\int_{x_n^0}^{x_n}U\,dx_n$. Then, equation (1.5) becomes:

$$\int_{x_n^0}^{x_n}\left\{\sum_{i,j=1}^{n}A_{ij}\left(g''\left(\frac{\partial U}{\partial x_n}+U\right)\left(\frac{\partial^2 U}{\partial x_i\,\partial x_n}+\frac{\partial U}{\partial x_i}\right)\left(\frac{\partial^2 U}{\partial x_j\,\partial x_n}+\frac{\partial U}{\partial x_j}\right)\right.\right.$$
$$\left.+g'\left(\frac{\partial U}{\partial x_n}+U\right)\left(\frac{\partial^3 U}{\partial x_i\,\partial x_j\,\partial x_n}+\frac{\partial^2 U}{\partial x_i\,\partial x_j}\right)\right)$$
$$\left.+\sum_{i=1}^{n}B_i g'\left(\frac{\partial U}{\partial x_n}+U\right)\left(\frac{\partial^2 U}{\partial x_i\,\partial x_n}+\frac{\partial U}{\partial x_i}\right)+Cg\left(\frac{\partial U}{\partial x_n}+U\right)\right\}dx_n=0.$$

If $A_{ij}+A_{ji}=0,\ B_i=\frac{\partial}{\partial x_n}(A_{in}+A_{ni})\qquad (i,j=1,\dots,n-1),$

$$B_n=\frac{\partial A_{nn}}{\partial x_n}+D(x_1,\dots,x_n),\ C=\frac{\partial D}{\partial x_n},$$

then the above equation becomes

$$\sum_{i=1}^{n-1}(A_{in}+A_{ni})g'\left(\frac{\partial U}{\partial x_n}+U\right)\left(\frac{\partial^2 U}{\partial x_i\,\partial x_n}+\frac{\partial U}{\partial x_i}\right) \tag{2.4}$$
$$+A_{nn}g'\left(\frac{\partial U}{\partial x_n}+U\right)\left(\frac{\partial^2 U}{\partial x_n^2}+\frac{\partial U}{\partial x_n}\right)+Dg\left(\frac{\partial U}{\partial x_n}+U\right)=0.$$

For the above equation the nonlinear superposition is given by

$$F=e^{-x_n}\int_{x_n^0}^{x_n} e^{-x_n}g^{-1}\left(C_1 g\left(\frac{\partial u}{\partial x_n}+u\right)+C_2 g\left(\frac{\partial v}{\partial x_n}+v\right)\right)dx_n+e^{-x_n}\varphi(x_1,\ldots,x_{n-1}),$$

where u, v are solutions of equation (2.4) and φ is an arbitrary function.

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