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OSCILLATING SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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1. A solution of a linear differential equation

$$y^{(m)}+\sum_{k=0}^{m-1} f_k(t)\, y^{(k)}=0 \tag{1}$$

will be called oscillatory if it changes sign for arbitrarily large t.

All the solutions of equation (1) form a linear vector space of dimension m. The bases of such a space for the third order equations

$$y'''+p(t)\, y''+q(t)\, y'+r(t)\, y=0 \tag{2}$$

have been investigated by W. R. Utz [1] and G. D. Jones [2].

Utz formulated the following theorem.

There are examples of equations (2) *for which there is a basis of the solution space with* 0, 1, 2, 3 *oscillatory elements.*

The example he gave is the equation

$$y'''-3y''+4y'-2y=0. \tag{3}$$

Jones obtained the following generalization of the constant coefficient case for the general third order self-adjoint equation

$$y'''+p(t)\, y'+\frac{1}{2}\, p'(t)\, y=0, \tag{4}$$

where $p(t)$, $p'(t)\in C(0, \infty)$. He proved the following statement.

If (4) *has an oscillatory solution, then its solution space has bases consisting of* 0, 1, 2 *or* 3 *oscillatory elements.*

In this paper we will extend the above results to differential equations of odd order.

2. We first note that if u and v are linearly independent functions on $A\subseteq \mathbf{R}$ then the functions u^k, $u^{k-1}v, \ldots, uv^{k-1}$, v^k $(k\in \mathbf{N})$ are also linearly independent.

Indeed, the equality

$$C_1 u^k + C_2 u^{k-1} v + \cdots C_k u v^{k-1} + C_{k+1} v^k = 0 \qquad (t \in A)$$

where $C_1, \ldots, C_{k+1}$ are arbitrary constants, can be written in the form

$$u^k \left(C_1 + C_2 \left(\frac{v}{u} \right) + \cdots + C_k \left(\frac{v}{u} \right)^{k-1} + C_{k+1} \left(\frac{v}{u} \right)^k \right) = 0,$$

which implies $C_1 = \cdots = C_{k+1} = 0$.

Suppose now that u and v are linearly independent solutions of the equation

$$(5) \qquad y'' + f(t)\, y = 0.$$

Then, according to the previous remark there exists a differential equation (E) of order $2n+1$, whose general solution is

$$y = \sum_{\nu=0}^{2n} C_\nu u^\nu v^{2n-\nu}.$$

Let $S(E)$ denote the solution space of (E). Then the sets $B_0, B_1, \ldots, B_{2n+1}$, defined by

$$B_0 = \{u^{2n},\ u^{2n-2} v^2, \ldots, v^{2n},\ (u+v)^{2n},\ (u+v)^{2n-2} v^2, \ldots, (u+v)^2 v^{2n-2}\},$$

$$B_\nu = \{u^{2n},\ u^{2n-2} v^2, \ldots, v^{2n},\ (u+v)^{2n},\ (u+v)^{2n-2} v^2, \ldots, (u+v)^{2\nu+2} v^{2n-2\nu-2},$$
$$u^{2n-1} v,\ u^{2n-3} v^3, \ldots, u^{2n+1-2\nu} v^{2\nu-1}\} \qquad (1 \le \nu \le n-1),$$

$$B_n = \{u^{2n},\ u^{2n-2} v^2, \ldots, v^{2n},\ u^{2n-1} v,\ u^{2n-3} v^3, \ldots, u v^{2n-1}\},$$

$$B_\nu = \{u^{2n},\ u^{2n-2} v^2, \ldots, u^{2\nu-2n} v^{4n-2\nu},\ u^{2n-1} v,\ u^{2n-3} v^3, \ldots, u v^{2n-1},$$
$$(u+v)^{2n-1} v,\ (u+v)^{2n-3} v^3, \ldots, (u+v)^{4n-2\nu+1} v^{2\nu-2n-1}\} \qquad (n+1 \le \nu \le 2n),$$

$$B_{2n+1} = \{u^{2n-1} v,\ u^{2n-3} v^3, \ldots, u v^{2n-1},\ (u+v)^{2n-1} v,\ (u+v)^{2n-3} v^3, \ldots,$$
$$(u+v)\, v^{2n-1},\ (u+v)\, u^{2n-1}\}$$

are bases of the solution space $S(E)$.

Moreover, if the equation (5) has an oscillatory solution then the basis B_k $(0 \le k \le 2n+1)$ has exactly k oscillatory elements. We have therefore proved the following

Theorem. *There exist examples of equations*

$$y^{(2n+1)} + \sum_{\nu=0}^{2n} f_\nu(t)\, y^{(\nu)} = 0 \qquad (n \in \mathbf{N})$$

for which there is a basis of the solution space with $0, 1, \ldots, 2n+1$ *oscillatory elements.*

Examples. 1. For $n=1$ our theorem reduces to the result of Utz.

2. Let $n=2$. It was shown in [3] that if u and v are linearly independent solutions of the equation (5) then the general solution of the equation

$$y^{(5)}+20f(t)\,y'''+30f'(t)\,y''+2\,(32f(t)^2+9f''(t))\,y' +4\,(16f(t)f'(t)+f'''(t))\,y=0$$

is given by

$$y=C_1u^4+C_2u^3v+C_3u^2v^2+C_4uv^3+C_5v^4,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

If $f(x)=1$, equation (5) becomes $y''+y=0$ and has oscillatory solutions $\cos t$ and $\sin t$. Hence, the solution space of the equation

$$y^{(5)}+20\,y'''+64\,y'=0$$

has bases with 0, 1, 2, 3, 4 or 5 oscillatory elements. They are

$$B_0=\{\cos^4 t,\ \cos^2 t\ \sin^2 t,\ \sin^4 t,\ (\cos t+\sin t)^4,\ (\cos t+\sin t)^2\sin^2 t\},$$

$$B_1=\{\cos^4 t,\ \cos^2 t\ \sin^2 t,\ \sin^4 t,\ (\cos t+\sin t)^4,\ \cos^3 t\ \sin t\},$$

$$B_2=\{\cos^4 t,\ \cos^2 t\ \sin^2 t,\ \sin^4 t,\ \cos^3 t\ \sin t,\ \cos t\ \sin^3 t\},$$

$$B_3=\{\cos^4 t,\ \cos^2 t\ \sin^2 t,\ \cos^3 t\ \sin t,\ \cos t\ \sin^3 t,\ (\cos t+\sin t)^3\sin t\},$$

$$B_4=\{\cos^4 t,\ \cos^3 t\ \sin t,\ \cos t\ \sin^3 t,\ (\cos t+\sin t)^3\sin t,\ (\cos t+\sin t)\sin^3 t\},$$

$$B_5=\{\cos^3 t\ \sin t,\ \cos t\ \sin^3 t,\ (\cos t+\sin t)^3\sin t,\ (\cos t+\sin t)\sin^3 t,\ (\cos t+\sin t)\cos^3 t\}.$$

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