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ANALOGIES BETWEEN DIFFERENTIAL AND DIFFERENCE EQUATIONS: SINGULAR SOLUTIONS OF DIFFERENCE EQUATIONS OF CLAIRAUT'S TYPE

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1. In paper [1] M. S. Klamkin considered the difference equation analogous to the generalized Clairaut's differential equation and obtained the following result:

Any sequence of the form

$$x_n = \sum_{k=0}^{m-1} a_k \binom{n}{k},$$

where $a_0, \ldots a_{m-1}$ are arbitrary constants such that $F(a_0, \ldots, a_{m-1}) = 0$, is a solution of the difference equation

$$F(z_0, z_1, \ldots, z_{m-1}) = 0,$$

where

$$z_r = \Delta^r x_n - n\,\Delta^{r+1} x_n + \frac{1}{2} n(n+1)\,\Delta^{r+2} x_n - \cdots$$
$$+ \frac{(-1)^{m-r-1}}{(m-r-1)!} n(n+1)\cdots(n+m-r-2)\,\Delta^{m-1} x_n.$$

The solution given by Klamkin corresponds to the general solution of the analogous differential equation. However, differential equations of Clairaut's type also possess singular solutions and it is therefore natural to expect the appearance of such solutions in the discrete analogue. There is no mention of such solutions in Klamkin's paper [1].

In this note we shall consider the following difference equation

$$x_n = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k!} n(n+1)\cdots(n+k-1)\,\Delta^k x_n + f(\Delta^m x_n), \tag{1.1}$$

where (x_n) is the unknown real sequence, $\Delta x_n = x_{n+1} - x_n$, $\Delta^k x_n = \Delta(\Delta^{k-1} x_n)$ $(k=2, 3, \ldots)$. It is the discrete analogue of the generalized Clairaut's differential equation

$$y = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k!} x^k y^{(k)} + f(y^{(m)}). \tag{1.2}$$

The general solution of the equation (1.2) is

$$y = \sum_{k=1}^{m} \frac{1}{k!} C_k x^k + f(C_m)$$

and the solution obtained by Klamkin for (1.1) reads

$$x_n = \sum_{k=1}^{m} \frac{1}{k!} C_k n^{(k)} + f(C_m), \tag{1.3}$$

where $n^{(k)} = n(n-1)\cdots(n-k+1)$, and $C_1, \ldots, C_m$ are, in both cases, arbitrary constants.

The analogy is, so far, perfect. However, as we shall soon see, this analogy breaks down when we consider singular solutions of equations (1.1) and (1.2).

2. Singular solutions of the equation (1.2) were considered in paper [2]. It was shown that the equation (1.2) is also satisfied by any function of the form

$$y = Y + \sum_{k=1}^{m-1} A_k x^k,$$

where $A_1, \ldots, A_{m-1}$ are arbitrary constants and Y is any particular solution of the equation

$$f'(y^{(m)}) = \frac{(-1)^m}{m!} x^m. \tag{2.1}$$

Similarly, solutions of (1.1) not contained in (1.3) can be obtained by solving the equation

$$\frac{f(\Delta^m x_{n+1}) - f(\Delta^m x_n)}{\Delta^{m+1} x_n} = \frac{(-1)^m}{m!} n(n+1)\cdots(n+m) \tag{2.2}$$

and checking the result by substituting into (1.1).

Equations (2.2) and (2.1) are clearly analogous. However, the solutions of the corresponding equations (1.1) and (1.2), to which they give rise, are not.

In order to see that, we shall consider two special cases of equations (1.1) and (1.2).

3. For $m=1$, $f(t)=t^2$, equations (1.1) and (1.2) become

$$x_n = n\,\Delta x_n + (\Delta x_n)^2 \tag{3.1}$$

and

$$y = xy' + (y')^2. \tag{3.2}$$

The singular solution of (3.2) is

$$y = -\frac{1}{4}x^2,$$

whereas it is readily verified that the corresponding (singular) solution of (3.1) is given by

$$x_n = -\frac{1}{4}n^2 + \frac{1}{16} + 4A^2 + A(-1)^n, \tag{3.3}$$

where A is an arbitrary constant.

Hence, the singular solution of (3.1) is not a fixed sequence, but a family of sequences defined by (3.3).

Remark. The equation (3.1) was considered in [3], and the sinqular solution (3.3) is given in [4].

For $m=2$, $f(t)=t^2$ equations (1.1) and (1.2) become

$$x_n = n\,\Delta x_n - \frac{1}{2}n(n+1)\,\Delta^2 x_n + (\Delta^2 x_n)^2 \tag{3.4}$$

and

$$y = xy' - \frac{1}{2}x^2 y'' + (y'')^2. \tag{3.5}$$

The singular solution of (3.5) is (see [2])

$$y = \frac{1}{48}x^4 + Ax,$$

where A is an arbitrary constant.

On the other hand, the corresponding solution of (3.4) is

$$x_n = \frac{1}{48}n^4 - \frac{1}{12}n^2 + \frac{1}{64} + An + 16B^2 + B(-1)^n,$$

where A and B are arbitrary constants.

In the general case, the singular solution of the equation (1.2) contains $m-1$ arbitrary constants. The above examples suggest that the singular solution of (1.1) contains m arbitrary constants.

4. The geometrical relation between the general and the singular solution of (3.2) is well known. Namely, the singular solution of (3.2) is the envelope of the family which defines the general solution of that equation.

It is interesting to note that some sort of analogy is preserved in the discrete analogue, i.e. the equation (3.1). In fact, for each fixed sequence (S) of the form (3.3) there exists a family of sequences of the form

$$x_n = Cn + C^2 \qquad (C \text{ arbitrary constant}) \tag{4.1}$$

such that (S) is the discrete envelope of that family.

For instance, put $A=0$ in (3.3). Then (3.3) is the discrete envelope of the family of sequences of the form (4.1) subjected to the condition that $-\left(2C+\frac{1}{2}\right)$ is a positive integer. Indeed, for each fixed $n_0 \in N$ there exists a constant C_0 $\left(C_0=-\frac{1}{4}(2n_0+1)\right)$ such that

$$C_0 n + C_0^2 = -\frac{1}{4}n^2 + \frac{1}{16}$$

and

$$\Delta\left(C_0 n + C_0^2\right) = \Delta\left(-\frac{1}{4}n^2 + \frac{1}{16}\right),$$

both for $n=n_0$.

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