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ON A CLASS OF SENTENTIAL FUNCTIONS

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Every n-ary ($n=0, 1, 2, \ldots$) truth function f of the two-valued sentential algebra (with $\top$, $\bot$ as constants) satisfying the condition

$$(*)\qquad f(\top, \top, \ldots, \top)=\top,$$

can be built using $\top$, $\wedge$ and $\Rightarrow$ only (the converse is obvious). This is proved by induction on n. For if $n=0$, f is $\top$, and if $n>0$, the proposition follows from

$$f(p_1, \ldots, p_n)=$$
$$\begin{cases} f_\bot(p_1, \ldots, p_{n-1})\Rightarrow p_n)\Rightarrow(f_\top(p_1, \ldots, p_{n-1})\wedge p_n), & \text{if } f_\bot(\top, \ldots, \top)=\top \\ ((\neg f_\bot(p_1, \ldots, p_{n-1})\Rightarrow p_n)\Rightarrow p_n)\Rightarrow(f_\top(p_1, \ldots, p_{n-1})\wedge p_n), & \\ & \text{if } f_\bot(\top, \ldots, \top)=\bot, \end{cases}$$

where the functions $f_\top$ and $f_\bot$ are defined by $f_\top(p_1, \ldots, p_{n-1})= f(p_1, \ldots, p_{n-1}, \top)$ and $f_\bot(p_1, \ldots, p_{n-1})=f(p_1, \ldots, p_{n-1}, \bot)$, because, by induction hypothesis, it holds for $f_\top(p_1, \ldots, p_{n-1})$ and *exactly one* of $f_\bot(p_1, \ldots, p_{n-1})$, $\neg f_\bot(p_1, \ldots, p_{n-1})$. As a simple corollary, *any* truth function can be conctructed this way, *at most one* use of negation.

There is a slight generalization of this result to the case of finite many-valued sentential algebra. Consider such an algebra with $E=\{1, 2, \ldots, n\}$ be a set of truth values ($n\geqslant 2$) and let, for some $l\leqslant s<n$, $D=\{1, 2, \ldots, s\}$ be a set of designated elements. Here, some truth functions have the following property, analogous to $(*)$:

(P) the restriction of the function to the domain D is itself an operation on D,

These are some of them, for example:

(1) maximum and minimum;

(2) $c_l(x_1, \ldots, x_m)=$

$$\begin{cases} s, & \text{if } (x_1, \ldots, x_m)\in D^m \\ l, & \text{otherwise} \end{cases}\qquad (l=s+1, \ldots, n;\quad m=1, 2, \ldots);$$

(3) $j_{k_1 \ldots k_m}(x_1, \ldots, x_m) =$

$$\begin{cases} l, \text{ if } (x_1, \ldots, x_m) = (k_1, \ldots, k_m) \\ s, \text{ if } (x_1, \ldots, x_m) \neq (k_1, \ldots, k_m) \text{ and } (x_1, \ldots, x_m) \in D^m \\ n, \text{ otherwise} \end{cases}$$

$$((k_1, \ldots, k_m) \in E^m, \quad m = 1, 2, \ldots).$$

It is interesting that the functions (1), (2) and (3), together with the elements of D as constants, constitute a *basis* for the class of truth functions with the property (P). For if f is in the class, it can be expressed as

$$f(x_1, \ldots, x_m) = \min \{\max (K_l, j_{k_1 \ldots k_m}(x_1, \ldots, x_m)) \mid f(k_1, \ldots, k_m) = l\}.$$

where

$$K_l = \begin{cases} l, \text{ if } l \in D \\ c_l\ (x_1, \ldots, x_m), \text{ if } l \notin D. \end{cases} \tag{4}$$

To prove this let $(x_1, \ldots, x_m)\ (e_1, \ldots, e_m)$ be an arbitrary m-tuple of elements of E. The proof splits in two cases.

a) $f(e_1, \ldots, e_m) = d \in D$. Then the "disjunct" $\max (K_d, j_{e_1 \ldots e_m}(x_1, \ldots, x_m)) = \max (d, l) = \text{d}$, by (4) and (3). All other "disjuncts" $D(x_1, \ldots, x_m)$ are of the form $\max (K_{f(k_1, \ldots, k_m)}, j_{k_1 \ldots k_m}(x_1, \ldots, x_m))$ with $(k_1, \ldots, k_m) \neq (e_1, \ldots, e_m)$, so $D(e_1, \ldots, e_m) \geqslant j_{k_1 \ldots k_m}(e_1, \ldots, e_m) \geqslant s$, by (3); hence, the minimum of all "disjuncts" is d.

b) $f(e_1, \ldots, e_m) = e \notin D$. Then, since f has the property (P), $(e_1, \ldots, e_m) \notin D^m$. Therefore, using (4), (3) and (2), $\max (K_e, j_{e_1 \ldots e_m}(x_1, \ldots, x_m))$ has the value $c_e(e_1, \ldots, e_m) = e$. For all other "disjuncts" $D(x_1, \ldots, x_m)$ it follows by (3) that

$D(e_1, \ldots, e_m) \geqslant j_{k_1 \ldots k_m}(e_1, \ldots, e_m) = n$ (since $(k_1, \ldots, k_m) \neq (e_1, \ldots, e_m)$),
i. e. the minimum of all "disjuncts" is e.

The above defined basis is not finite, so the next problem is to look for a finite one. The problem is settled by the relations

$$c_r(x_1, \ldots, x_m) = \max (c_r(x_1), \ldots, c_r(x_m)) \qquad (r = s+1, \ldots, n), \tag{5}$$

$$j_{k_1 \ldots k_m}(x_1, \ldots, x_m) = \max \{j_{qt}(x_q, x_t) \mid q, t \in \{k_1, \ldots, k_m\}\} \tag{6}$$

(especially, for $m = 1$ we get $j_k(x) = j_{kk}(x, x)$),

because it follows from them that the following functions constitute a finite basis for the class of truth functions with the property (P), namely:

1° constants $1, 2, \ldots, s$;

2° $c_r(x) \quad r = s+1, \ldots, n$;

3° $j_{qt}(x, y), \quad q, t \in E$[1].

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[1] Functions c_r and j_{qt} are, of course, defined as in (2) and (3), resp.

To get this it remains to prove (5) and (6). For (5) note that the right side $R(x_1, \dots, x_m)$ of (5) has the property (P) and also

$$s \leqslant R(x_1, \dots, x_m) \leqslant r, \tag{7}$$

by (2). So if $(x_1, \dots, x_m) \in D^m$, then $R(x_1, \dots, x_m) = s$ and if $(x_1, \dots, x_m \notin D^m$, then some $x_i \notin D$ $(l \leqslant i \leqslant m)$, hence $R(x_1, \dots, x_m) \geqslant c_r(x_i) = r$ by (2), i. e. $R(x_1, \dots, x_m) = r$, by (7).

To prove (6) first let $(x_1, \dots, x_m) = (k_1, \dots, k_m)$. Then the right side $S(x_1, \dots, x_m)$ of (6) is obviously l by (3). Secondly, if $(x_1, \dots, x_m) = (b_1, \dots, b_m) \neq (k_1, \dots, k_m)$ and $(b_1, \dots, b_m) \in D^m$, then some $b_i \neq k_i$ $(l \leqslant i \leqslant m)$, so $S(b_1, \dots, b_m) \geqslant j_{k_i k_i}(b_i, b_i) = s$, by (3). Moreover, $S(b_1, \dots, b_m) \leqslant s$ since $S(x_1, \dots, x_m)$ has the property (P), hence $S(b_1, \dots, b_m) = s$. Finally, if $(x_1, \dots, x_m) = (c_1, \dots, c_m) \neq (k_1, \dots, k_m)$ and $(c_1, \dots, c_m) \notin D^m$, then some $c_q \neq k_q$ $(l \leqslant q \leqslant m)$ and some $c_t \notin D$ $(l \leqslant t \leqslant m$; the case $q = t$ included). It follows by (3) that $j_{k_q k_t}(c_q, c_t) = n$, hence $S(c_1, \dots, c_m) = n$.

REFERENCE

[1] J. B. Rosser, A. Turquette, *Many-Valued Logic*, Amsterdam, 1958

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