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A REMARK ON FILTER REGULARITY

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In this paper we give two statements on filter regularity.

Definition 1. 1. For a filter D over the set I we say that it is (α, β)-regular if and only if there is an $E \subset D$ such that:

1. $|E| = \beta$ ($|E|$ denotes the cardinality of E)
2. $X \subset E$ and $|X| \geqslant \alpha$ implies $\bigcap X = \varnothing$.

In this case we say that E is an (α, β)-regular family in D. If D is (ω, β)-regular, we say β-regular, if $|I| = \beta$ and D is β-regular we say that D is regular (this definition is from [2]).

The first statement is related to the following result of Magidor [3].

1.2. If there is an ω-regular, non-uniform ultrafilter on ω_1 which is not ω_1-regular then

$$\forall \alpha < \omega \quad 2^{\aleph_\alpha} = \aleph_{\alpha+1} \Rightarrow 2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$$

(we think that in the above, $\forall \alpha < \omega_1$ should stay instead of $\forall \alpha < \omega$).

Proposition 1. 3. *Given any two cardinals* α, β $(\alpha \leqslant \beta)$ *there is an ultrafilter* U *over* β *which is only* α*-regular.*

Proof. According to proposition 4.3.5. of [1] there is an α-regular ultrafilter A over α. We may suppose $\alpha < \beta$. We have $\alpha \subset \beta$; hence $A \subset S(\alpha) \subset S(\beta)$. ($S(\alpha)$ is the power set of α). Therefore A can be extended to an ultrafilter over β. Suppose that U is such an ultrafilter. U is α-regular. Suppose that U is γ-regular for some $\gamma > \alpha$, and E' is a γ-regular family in U. Let f be the mapping given by $f: E' \to U$, $f(e') = e' \cap \alpha$; $f(e') \in U$ for all $e' \in E'$. Let $E'' = f(E')$. Then $|E''| = \gamma$, for otherwise there would be some $X' \subset E'$ such that $|X'| \geqslant \omega$ and $|f(X')| = 1$. This means that there is a $u \in U$ such that $u = e' \cap \alpha = f(e')$ for all $e' \in X'$. Hence, $\bigcap X' \supset u \neq \varnothing$ contradicting the regularity of E'. Therefore $|E''| = \gamma$. Let $g: \alpha \to S(E'')$ such that $g(j) = \{e'' \mid e'' \in E'' \,\&\, j \in e''\}$; for all $j \in \alpha$ we have $|g(j)| < < \omega$; $\bigcup_{j \in \alpha} g(j) = E''$. It follows that $|E''| \leqslant \sum_{j \in \alpha} |g(j)| = \alpha$, contradicting $|E''| = \gamma > \alpha$. Therefore U is not γ-regular

Corollary 1. 4, *Replacing* α, β *in the above proposition by* ω, ω_1, *we obtain a non uniform* ω*-regular ultrafilter over* ω_1 *which is not* ω_1*-regular; thus, one can exclude the filter existence hypothesis from* 1.2. *obtaining,*

$$\forall \alpha<\omega,\ 2^{\aleph_\alpha}=\aleph_{\alpha+1}\ \Rightarrow\ 2^{\aleph_{\omega 1}}=\aleph_{\omega 1+1}.$$

To list the second statement we need the definition of product of ultrafilters.

Definition 1. 5. (Following [1])

Let D, E be ultrafilters over sets I, J, define $D\times E$ to be the set of all $Y\in S(I\times J)$ such that $\{j\in J:\{i\in I: <i, j>\in Y\}\in D\}\in E$.

Proposition 1. 6.

Let D, E *be uniform ultrafilters over* k, k^+ *then* $D\times E$ *and* $E\times D$ *are* (k, k^+)*-regular.*

Proof. Let $\beta\in k^+$; then $|\beta|\leqslant k$ and β can be written as a sequence of length $|\beta|$:

1. $\beta=\{\xi_0^\beta,\dots\ \xi_\nu^\beta,\dots\ |\nu<|\beta|\leqslant k\}$.

Let

2. $\sigma(\nu,\ \beta)=\{\xi_i^\beta\,|\,i<\nu<|\beta|\leqslant k\}$;

define $e_\xi=\{<\nu,\ \beta>|\,\xi\in\sigma(\nu,\ \beta\}$ for all $\xi\in k^+$, and

$$\mathcal{E}=\{e_\xi\,|\,\xi<k^+\}.$$

We shall prove that $\mathcal{E}$ is a (k, k^+)-regular family in $D\times E$. Let $e_\xi\in\mathcal{E}$ and let $\beta\geqslant\max(\xi,\ k)+1$. Then $\xi\in\beta$, so by 1. and 2. it follows that there is a $\nu<k$ such that $\xi\in\sigma(\nu,\ \beta)$. By 2., for all $\nu'\in[\nu,\ k)$, we have $\xi\in\sigma(\nu',\ \beta)$ and hence $\langle\nu',\ \beta\rangle\in e_\xi$. Thus, for all $\beta\in[\max(\xi,\ k)+1,\ k^+)$ there is a $\nu(\beta)$ such that $\{\langle\nu,\ \beta\rangle\,|\,\nu\in[\nu(\beta),\ k)\}=\sigma_\beta^\xi\subset e_\xi$. Hence, $e_\xi\supset\bigcup_{\beta\in[\max(\xi,\ k)+1,\ k+)}\sigma_\beta^\xi\in D\times E$. It follows that $\mathcal{E}\subset D\times E$. Let $\xi\neq\xi'$. We shall prove that $e_\xi\neq e_\xi'$. This implies

$$|\mathcal{E}|=k^+.$$

Let $\beta=\max(\xi,\ \xi')+1$; then ξ, $\xi'\in\beta$, and let following 1., $\xi=\xi_{\nu_\xi}^\beta$, $\xi'=\xi_{\nu_\xi}^\beta$, Suppose $\nu_\xi<\nu_\xi'$; then we have $\xi\in\sigma(\nu_\xi',\ \beta)$ and $\xi'\notin\sigma(\nu_\xi',\ \beta)$. This implies $\langle\nu_\xi',\ \beta\rangle\in e_\xi$ and $\langle\nu_\xi',\ \beta\rangle\notin e_\xi'$. Hence $e_\xi\supset e_\xi'$ and $|\mathcal{E}|=k^+$. Now suppose $X\subset\mathcal{E}$, $|X|\geqslant k$ and $\cap X\neq\varnothing$. Then there is a pair $\langle\nu,\ \beta\rangle\in k\times k^+$ such that $\langle\nu,\ \beta\rangle\in\cap X$; since

$$X=\{e_\xi\,|\,\xi\in X_0\subset k^+\,\&\,|X_0|\geqslant k\},$$

we have

$$\langle\nu,\ \beta\rangle\in\bigcap_{\xi\in X_0}e_\xi \text{ or, equivalently,}$$

for all $\xi\in X_0$, $\xi\in\sigma(\nu,\ \beta)$; hence

$$|X_0|\leqslant|\sigma(\nu,\ \beta)|=|\nu|\leqslant\nu<|\beta)\leqslant k,$$

contradicting

$$|X_0|\geqslant k;\ \text{therefore}\ \cap X=\varnothing.$$

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