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ON BASES AND PROJECTIONS IN NON-ARCHIMEDEAN BANACH SPACES

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1. Introduction

In [3, 5], it has been shown that a non-archimedean (n.a). Banach space of countable type has a basis. Very recently, the authors [1] have obtained a necessary and sufficient condition for a general n.a. Banach space (countable or uncountable type) to have a basis and derived certain criterion for a basis to be orthogonal. In this paper, it has been observed that there is close connection between the existence of bases and projections. In fact if there is a basis for a closed linear subspace F of a n.a. Banach space E, a very general condition allows to define a projection of E on F.

2. Notations and Terminology

Let K be a n.a. non-trivial valued field which is complete under the metric of valuation of rank one. Throughout, by E we shall mean a n.a. Banach space over the field K with n.a. norm $\| \, \|$. Let E', E'', E''' denote the duals of E, E' E'' respectively. Let J, J' be the natural embeddings of E into E'' and E' into E'''. For general properties of n.a. Banach spaces we refer to [2, 4, 6].

Let $X \subset E \setminus \{0\}$ be any system of vectors. We may write

$$X = \{x_\lambda : \lambda \in \Lambda\},$$

where Λ is an index set of any cardinality. Let Σ be the set of all finite subsets of Λ directed by inclusion.

A system X is said to be summable to x in E if $\lim\limits_{\sigma} \sum\limits_{\lambda \in \sigma} x_\lambda = x$, $\sigma \in \Sigma$, where $\lim\limits_{\sigma} y_\sigma$ denotes the limit of a net $\{y_\sigma : \sigma \in \Sigma\}$ in E. Further, a system X

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is a basis for E if to each x in E there exists a unique system $\{\alpha_\lambda : \lambda \in \Lambda\}$ of scalars such that $\{\alpha_\lambda x_\lambda : \lambda \in \Lambda\}$ is summable to x i.e.

$$x = \lim_{\sigma} \sum_{\lambda \in \sigma} \alpha_\lambda x_\lambda, \quad \sigma \in \Sigma \tag{2.1}$$

Clearly, with each basis $\{x_\lambda\}$ there is associated a unique family $\{f_\lambda\}$ of linear functionals on E such that $f_\lambda(x) = \alpha_\lambda$, where x is given by (2.1). Thus, without ambiguity, we may write a basis $\{x_\lambda\}$ as $\{x_\lambda, f_\lambda\}$ as and when we need so. If for a basis $\{x_\lambda, f_\lambda\}$ the family $\{f_\lambda\}$ is in E', then the basis is said to be Schauder.

A double system $\{x_\lambda, f_\lambda\}$, $x_\lambda \in E$, $f_\lambda \in E'$ is called a biorthogonal system if $f_\mu(x_\lambda) = \delta_{\lambda\mu}$, where $\delta_{\lambda\mu}$ denotes the Kronecker symbol. Let $\{U_\sigma\}$ be a family of linear operators on E defined by

$$U_\sigma x = \sum_{\lambda \in \sigma} f_\lambda(x) x_\lambda, \quad x \in E, \ \sigma \in \Sigma.$$

Obviously, the operators U_σ are continuous projections of E with the property $U_\sigma U_\tau = U_{\sigma \cap \tau}$.

By a projection on E we mean a linear transformation $P : E \to E$ such that $P^2 = P$.

3. Projections on E

Theorem 3.1. *Let $\{y_\lambda : \lambda \in \Lambda\}$ be a basis system* in E. If there exists a system $\{f_\lambda : \lambda \in \Lambda\}$ in E' such that $f_\lambda(y_\mu) = \delta_{\lambda\mu}$ and $\{f_\lambda(x) y_\lambda\}$ is summable in E for all x in E, then P given by*

$$Px = \lim_{\sigma} \sum_{\lambda \in \sigma} f_\lambda(x) y_\lambda, \tag{3.1}$$

defines a projection of E on $\overline{sp}\{y_\lambda\}$. Conversely, if K is spherically complete and P is a projection of E on $\overline{sp}\{y_\lambda\}$, then there exists a unique system $\{f_\lambda : \lambda \in \Lambda\}$ in E' such that $f_\lambda(y_\mu) = \delta_{\lambda\mu}$ and (3.1) *holds.*

Proof. The proof of first part is simple and is omitted. For the converse, let $\{g_\lambda : \lambda \in \Lambda\}$ be the associated biorthogonal family to $\{y_\lambda : \lambda \in \Lambda\}$. Since K is spherically complete, by Hahn-Banach theorem (see [4], p. 8) there exists an extension g'_λ of g_λ $(\lambda \in \Lambda)$ to the whole space E. Obviously, $g'_\lambda \in E'$.

Let $P^* : \{\overline{sp}\{y_\lambda\}\}' \to E'$ be defined by

$$P^* g(x) = g(Px), \quad g \in \{\overline{sp}\{y_\lambda\}\}', \ x \in E.$$

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* $\{y_\lambda : \lambda \in \Lambda\}$ is said to be a basis system in E if $\{y_\lambda\}$ is a basis for $\overline{sp}\{y_\lambda\}$.

Take $f_\lambda = P^* g'_\lambda \in E'$. Then $f_\lambda(y_\mu) = \delta_{\lambda\mu}$. Since $\{y_\lambda, g_\lambda\}$ is a basis for $\overline{sp}\{y_\lambda\}$, we have

$$Px = \lim \sum_{\lambda\in\sigma} g_\lambda (Px)\, y_\lambda$$

$$= \lim_\sigma \sum_{\lambda\in\sigma} f_\lambda(x)\, y_\lambda,\ x\in E.$$

The uniqueness of the family $\{f_\lambda\}$ follows obviously.

As an immediate consequence of the first part in Theorem 3.1, we have.

Theorem 3.2. *Let $\{x_\lambda, f_\lambda\}$ be a basis for E. If $\{F(f_\lambda)\, x_\lambda : \lambda\in\Lambda\}$ is summable in E for all F in E'', then $J(E)$ is complemented** in E''.*

The converse of theorem 3.2 is available in

Theorem 3.3. *Let $\{x_\lambda. f_\lambda\}$ be a basis for E and P a projection of of unit norm of E'' on $J(E)$. If for every f in E' there is unique $\mathfrak{F}$ in E''' such that $\|\mathfrak{F}\| = \|f\|$and $f(x) = \mathfrak{F}(Jx)$ for all x in E, then $\{F(f_\lambda)\, x_\lambda : \lambda\in\Lambda\}$ is summable in E for all F in E''.*

Proof. Since $\{x_\lambda, f_\lambda\}$ is a basis for E, we have

$$J^{-1}PF = \lim_\sigma \sum_{\lambda\in\sigma} f_\lambda(J^{-1}PF)\, x_\lambda$$

$$= \lim_\sigma \sum_{\lambda\in\sigma} J' f_\lambda(PF)\, x_\lambda$$

$$= \lim_\sigma \sum_{\lambda\in\sigma} P^* J' f_\lambda(F)\, x_\lambda,\ F\in E'',$$

where P^* defined as in Theorem 3.1. To establish the theorem it is enough to show that $P^* J' f_\lambda = J' f_\lambda$.

Let F_λ be the restriction of $J' f_\lambda$ to $J(E)$. Then $F_\lambda(Jx) = J' f_\lambda(Jx) = f_\lambda(x)$, $x\in E$. Therefore

$$\|F_\lambda\| = \sup\left\{\frac{\|F_\lambda(Jx)\|}{\|Jx\|} : Jx \neq 0\right\}$$

$$= \sup\left\{\frac{\|f_\lambda(x)\|}{\|Jx\|} : Jx \neq 0\right\}$$

$$\geqslant \sup\left\{\frac{\|f_\lambda(x)\|}{\|x\|} : x \neq 0\right\}$$

since $\|Jx\| \leqslant \|x\|$. Thus

$$\|F_\lambda\| \geqslant \|f_\lambda\| \geqslant \|J' f_\lambda\|.$$

** A subspace F of E is said to be complemented in E if there is a projection from E onto F.

Further $P^* J' f_\lambda \in E'''$ and $P^* J' f_\lambda (Jx) = J' f_\lambda (PJx) = J' f_\lambda (Jx) = F_\lambda (Jx)$, for all x in E. Consequently, $P^* J' f_\lambda$ is an extension of F_λ to E'' and so

$$\| F_\lambda \| \leqslant \| P^* J' f_\lambda \| \leqslant \| P^* \| \, \| J' f_\lambda \| \leqslant \| J' f_\lambda \|,$$

because $\| P^* \| \leqslant \| P \| = 1$. Hence, $\| P^* J' f_\lambda \| = \| J' f_\lambda \|$. Since $f_\lambda (x) = J' f_\lambda (Jx) = P^* J' f_\lambda (Jx)$ for all x in E, it follows that $P^* J' f_\lambda = J' f_\lambda$.

4. Existence of basis in E

Theorem 4.1. *It $\{x_\lambda : \lambda \in \Lambda\}$ is a basis for E, then there exists a positive constant α such that*

$$\text{dist. } (S_\sigma, E_\sigma) \geqslant \alpha, \quad \sigma \in \Sigma,$$

where

(4.1) $$S_\sigma = \{x : x \in sp\, \{x_\lambda : \lambda \in \sigma\}, \ \| x \| = 1\}$$

(4.2) $$E_\sigma = \overline{sp}\, \{x_\lambda : \lambda \in \Lambda \sim \sigma\}$$

(4.3) $$\text{dist}\, \{S_\sigma, E_\sigma) = \inf \| x - y \| : x \in S_\sigma, \ y \in E_\sigma\}.$$

The converse holds provided $\| E \| = | K |$.

Proof. Since $\| x \| \leqslant \sup\limits_\sigma \| U_\sigma x \| < \infty$ for all x in E, we have ([4], p. 75)

$$1 \leqslant \sup \left\{ \frac{\sup\limits_\sigma \| U_\sigma x \|}{\| x \|} : x \neq 0 \right\} = \sup_\sigma \| U_\sigma \| < \infty .$$

Take

$$\alpha = \frac{1}{\sup\limits_\sigma \| U_\sigma \|},$$

so that $0 < \alpha \leqslant 1$. Since each U_σ is a continuous projection, $(I - U_\sigma)$ is closed and hence $E_\sigma \subset (I - U_\sigma) E$ Therefore, S_σ being the unit sphere of $U_\sigma E$, we have dist $(S_\sigma, E_\sigma) = \inf \{\| x - y \| : x \in U_\sigma E, \ y \in E_\sigma, \ \| x \| = 1\}$

$$\geqslant \inf \{\| x - y \| : x \in U_\sigma E, \ y \in (I - U_\sigma) E, \ \| x \| = 1\}$$

$$\geqslant \alpha \inf \{\| U_\sigma (x - y) \| : x \in U_\sigma E, \ y \in (I - U_\sigma) E, \ \| x \| = 1\},$$

since $\| U_\sigma (x - y) \| \leqslant \| U_\sigma \| \, \| x - y \| \leqslant \alpha^{-1} \| x - y \|$. Further it, may be noted that $U_\sigma (x - y) = x$. Hence

$$\text{dist } (S_\sigma, E_\sigma) \geqslant \alpha.$$

The proof of the converse part is simple and the details are omitted.

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