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THE DISTRIBUTION OF VALUES OF SOME MULTIPLICATIVE FUNCTIONS

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1. Introduction

Let us consider the following examples of multiplicative arithmetical functions:

a) Sum of divisors function $\sigma(n)=\sum_{d\mid n} d$, $\sigma(p^a)=1+p+\cdots+p^a$ (p denotes a prime number throughout the paper, $\sum_{d\mid n}$ summation over all divisors on n).

b) Euler's totient function $\varphi(n)=\sum_{m<n,\ (m,\ n)=1} 1=n\sum_{d\mid n}\mu(d)/d=$ the number of integers less than n which are relatively prime to n. Here $\mu(n)$ as usual stands for the Möbius function, and $\varphi(p^a)=p^a-p^{a-1}$.

c) Dedekind's function $\psi(n)=n\prod_{p\mid n}(1+1/p)=n\sum_{d\mid n}\mu^2(d)/d$. Here $\prod_{p\mid n}$ denotes the product over all different prime divisors of n and $\psi(p^a)=p^a+p^{a-1}$.

d) Unitary analogue of the sum of divisors function $\sigma^*(n)=\sum_{d\mid n,(d,\ n/d)=1} d$, so that $\sigma^*(n)$ is the sum of divisors d of n for which d and n/d are relatively prime (such divisors d are called unitary divisors of n). We have $\sigma^*(p^a)=p^a+1$.

e) Unitary analogue of the totient function: $\varphi^*(n)=n\sum_{d\mid n,\ (d,\ n/d)=1}(-1)^{\omega(d)}/d$ where $\omega(n)$ is the number of distinct prime factors of n, $\varphi^*(p^a)=p^a-1$.

For a more detailed account of $\sigma(n)$ and $\varphi(n)$ consult [7], for $\psi(n)$ see [8], and for $\sigma^*(n)$ and $\varphi^*(n)$ see [3]. All of the above mentioned functions have the common property that they are multiplicative, positive and that

$$f(p^k)=p^k+a_{1,\ k}\,p^{k-1}+a_{2,\ k}\,p^{k-2}+\cdots+a_{k,\ k}$$

where $|a_{i,\ k}|\leq 1$ for all k and $i=1, 2, \ldots, k$. Therefore, we may define a general class of arithmetical functions D which contains all of the mentioned functions as follows:

Definition: A multiplicative function $f(n)$ belongs to the class D if for every prime p and every natural number k there exist numbers $a_{1,k}, a_{2,k}, \ldots, a_{k,k}$ such that

$$f(p^k) = p^k + a_{1,k} p^{k-1} + a_{2,k} p^{k-2} + \cdots + a_{k,k} \tag{1}$$

where $-1 \leq a_{i,k} \leq K$ for some non-negative K and all k and $i = 1, 2, \ldots, k$.

From this definition it is obvious that $f(n)$ is strictly positive and that $f(n)$ is a natural number if the $a_{i,k}$'s are integers (if the $a_{i,k}$'s were allowed to take smaller integer values than -1 then $f(n)$ would not always be positive).

For every arithmetical function $f(n)$ we may define a new function $\bar{f}(n)$ as

$$\bar{f}(n) = \sum_{f(m)=n} 1, \tag{2}$$

that is, as the number of solutions of the equation $f(m) = n$ in m, if n is given. Then $N(x) = \sum_{n \leq x} \bar{f}(n) = \sum_{f(m) \leq x} 1$ is the number of integers m from which $f(m) \leq x$.

The main purpose of this paper is to investigate the asymptotic formula for $N(x)$ when $f(n)$ belongs to the class D. Since from (1) we see that $f(n)$ is in a certain sense about the same order of magnitude as n, we may suppose that $N(x)$ will behave asymptotically as Cx for a suitable positive constant C. Theorem 2 shows that this is indeed so, giving a more precise result; the method of proof used there originated with Paul T. Bateman, [1], who investigated the distribution of values of the Euler function $\varphi(n)$. One might because of (1) also expect that as $x \to \infty$ $\sum_{n \leq x} f(n) \sim Dx^2$ (where D is a suitable positive constant) since $\sum_{n \leq x} n \sim x^2/2$ as $x \to \infty$. This is not difficult to obtain; if we set $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$, $G(s) = F(s)/\zeta(s-1)$ then using (1) and $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ (valid for $\operatorname{Re} s > 1$) we see that the abscissa of absolute convergence of $G(s)$ equals 1 and therefore a classical convolution argument (see [7] for the corresponding results concerning $\sigma(n)$ and $\varphi(n)$) gives for every $\varepsilon > 0$

$$\sum_{n \leq x} f(n) = \frac{G(2)}{2} x^2 + O(x^{1+\varepsilon}), \tag{3}$$

and additional information about $f(n)$ may lead to improvements of the error term.

2. Statement and proof of theorems

Theorem 1. *If $f(n)$ belongs to the class D then there exist positive numbers C_1, C_2 and a natural number n_1 such that*

$$f(n) \leq C_1 n (\log\log n)^K \qquad \text{for } n \geq n_1 \tag{4}$$

$$f(n) \geq C_2 m/\log\log m \qquad \text{for } m > 1,\ n = 2^k m,\ m \text{ odd} \tag{5}$$

where K is the constant such that $a_{i,k} \leq K$ for all k and $i = 1, 2, \ldots, k$ and $a_{i,k}$ are the numbers appearing in (1).

Theorem 2, *If $f(n)$ belongs to the class D and $a_{1,1}$ is an integer then*

$$N(x) = \sum_{f(n) \leq x} 1 = Cx + O(x \cdot \exp(-d \log^{3/8-\varepsilon} x)) \tag{6}$$

where d and ε are arbitrary positive numbers, $C = \lim_{s \to 1+0} (s-1) H(s)$, $H(s) = \sum_{n=1}^{\infty} (f(n))^{-s}$.

Theorem 3. *If $f(n)$ belongs to the class D then*

$$\sum_{n \leq x}' \frac{1}{\log f(n)} = \frac{x}{\log x} \left(1 + O\left(\frac{\log\log\log x}{\log x}\right)\right) \tag{7}$$

where the dash ' denotes summation over all n for which $f(n) > 1$.

Proof of theorem 1. If $p^k \| n$ means that p^k divides n and that p^{k+1} does not, then since $f(n)$ is multiplicative we have by (1)

$$f(n) = \prod_{p^k \| n} f(p^k) \leq \prod_{p^k \| n} \left(p^k + K\frac{p^k - 1}{p-1}\right) \leq \prod_{p^k \| n} \left(p^k + \frac{Kp^k}{p-1}\right) = n \prod_{p \mid n} \left(1 + \frac{K}{p-1}\right).$$

$$\log \prod_{p \mid n} \left(1 + \frac{K}{p-1}\right) = \sum_{p, n} \log \left(1 + \frac{K}{p-1}\right) \leq K \sum_{p \mid n} \frac{1}{p-1} \leq K \sum_{p \leq p_m} \frac{1}{p-1},$$

where $m = \omega(n)$ denotes the number of dictinct prime divisors of n and p_m denotes m-th prime number.

Since $\sum_{p \leq x} \frac{1}{p-1} - \sum_{p \leq x} \frac{1}{p} = \sum_{p \leq x} \frac{1}{p^2 - p} = O(1)$ and (see [7])

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1)$$

it follows that for some constant $B > 0$ and $n \geq n_1$ $\log \prod_{p \mid n} \left(1 + \frac{K}{p-1}\right) \leq$ $\leq K \log(B \log p_m)$.

From the elementary estimate $p_n \leq n^{3/2}$ valid for $n \geq 3$ we obtain

$$\log \prod_{p \mid n} \left(1 + \frac{K}{p-1}\right) \leq K \log\left(\frac{3}{2} B \log m\right).$$

Using the elementary fact that $n \geq \prod_{p \mid n} p$ we have $\log n \geq \omega(n) \log 2$, and so for $n \geq 5$ $\log m = \log \omega(n) \leq \log\log n - \log\log 2 \leq 2 \log\log n$ which proves (4) with $C_1 = (3B)^K$.

To prove (5) note that by (1) $f(p^k)\geq p^k-p^{k-1}-p^{k-2}-\cdots-p-1$ so that $f(p^k)=1$ possibly only for $p=2$, otherwise $f(p^k)>1$ and we have

$$(8)\quad f(n)=f(2^k)f(m)\geq \prod_{p^k\| m}\left(p^k-\frac{p^k-1}{p-1}\right)=\prod_{p^k\| m}\frac{p^{k+1}-2p^k+1}{p-1}\geq m\prod_{p\mid m}\frac{p-2}{p-1}.$$

Since for $0\leq x\leq 1/2$

$$\log\frac{1}{1-x}\leq x+x^2$$

then

$$\log\prod_{p\mid m}\left(1-\frac{1}{p-1}\right)^{-1}=\sum_{p\mid m}\log\left(1-\frac{1}{p-1}\right)^{-1}\leq\sum_{p\mid m}\left(\frac{1}{p-1}+\frac{1}{(p-1)^2}\right)\leq$$
$$\leq\log(C_3\log\log m)+O(1)\leq\log(C_4\log\log m)$$

for $m>1$ and C_4 large enough, so that with $C_2=C_4^{-1}$

$$\prod_{p\mid m}\left(1-\frac{1}{p-1}\right)^{-1}=\prod_{p\mid m}\frac{p-1}{p-2}\leq C_2^{-1}\log\log m$$

which combined with (8) proves (5).

Sharper estimates of $\sum_{p\leq x}1/p$ and p_n would lead to explicit values of n_1, C_1 and C_2, but C_1 and C_2 would still depend on K. Taking $n=p_1p_2\ldots p_k$ where $2=p_1<p_2<\cdots<p_k$ are the first k primes it is seen that the bounds of (4) and (5) are attained.

To prove theorem 2, the following lemma is needed:

Lemma 1. *If $f(n)$ belongs to the class D and $H(s)=\sum_{n=1}^{\infty}(f(n))^{-s}$ then*

$$(9)\ H(s)=(1+(f(2))^{-s}+(f(2^2))^{-s}+\cdots)\prod_{p>2}(1-(p+a_{1,1})^{-s})^{-1}\prod_{p>2}(1+a(p,s))$$

where $\prod_{p>2}(1+a(p,s))$ is absolutely convergent for $\sigma=\mathrm{Re}\,s>1/2$.

Proof. Since $f(n)$ is multiplicative we have

$$H(s)=\prod_p(1+(p+a_{1,1})^{-s}+(p^2+a_{1,2}p+a_{2,2})^{-s}+(p^3+a_{1,3}p^2+a_{2,3}p+a_{3,3})^{-s}+$$
$$+\cdots)=(1+(f(2))^{-s}+(f(2^2))^{-s}+\cdots)\prod_{p>2}(1+(p+a_{1,1})^{-s}+(p^2+a_{1,2}p+$$
$$+a_{2,2})^{-s}+\cdots)$$

so that we may set

$$1+a(p,s)=(1+(p+a_{1,1})^{-s}+(p^2+a_{1,2}p+a_{2,2})^{-s}+\cdots)(1-(p+a_{1,1})^{-s})$$

and therefore

$$a(p,s)=\sum_{n=2}^{\infty}\{(p^n+a_{1,n}p^{n-1}+\cdots+a_{n,n})^{-s}-(p+a_{1,1})^{-s}(p^{n-1}+a_{1,n-1}p^{n-2}+$$
$$+\cdots+a_{n-1,n-1})^{-s}\}=\sum_{n=2}^{\infty}A(p,s)/B(p,s).$$

$$|A(p,s)|=|(p+a_{1,1})^s(p^{n-1}+a_{1,n-1}p^{n-2}+\cdots+a_{n-1,n-1})^s-(p^n+a_{1,n}p^{n-1}+$$
$$+\cdots+a_{n,n})^s|\leq(p+K)^\sigma(p^{n-1}+Kp^{n-2}+\cdots+K)^\sigma+(p^n+Kp^{n-1}+\cdots\ K)^\sigma=$$
$$(p+K)^\sigma\left(p^{n-1}+K\frac{p^{n-1}-1}{p-1}\right)^\sigma+\left(p^n+K\frac{p^n-1}{p-1}\right)^\sigma\leq 2(1+K)^\sigma(p+K)^\sigma p^{(n-1)\sigma}.$$

$$|B(p,s)|=|(p^n+a_{1,n}p^{n-1}+\cdots+a_{n,n})^s(p+a_{1,1})^s(p^{n-1}+a_{1,n-1}p^{n-2}+\cdots+$$
$$+a_{n-1,n-1})^s|\geq(p-1)^\sigma(p^n-p^{n-1}-\cdots-p-1)^\sigma(p^{n-1}-p^{n-2}-\cdots-p-1)^\sigma$$
$$=(p-1)^\sigma\left(p^n-\frac{p^n-1}{p-1}\right)^\sigma\left(p^{n-1}-\frac{p^{n-1}-1}{p-1}\right)^\sigma=$$
$$\{(p^{n+1}-2p^n+1)(p^n-2p^{n-1}+1)(p-1)^{-1}\}^\sigma\geq\{p^{2n-1}(p-2)^2(p-1)^{-1}\}^\sigma$$
$$\geq\left(\frac{p-2}{4}\cdot p^{2n-1}\right)^\sigma.$$

Therefore, we have

$$|a(p,s)|\leq 2\sum_{n=2}^{\infty}(4+4K)^\sigma\left(\frac{p+K}{p-2}\right)^\sigma p^{-n\sigma}=2(4+4K)^\sigma\left(\frac{p+K}{p-2}\right)^\sigma(p^{2\sigma}-p^\sigma)^{-1},$$

and so $\prod_{p>2}(1+a(p,s))$ is absolutely convergent for $\mathrm{Re}\, s>1/2$ since $\left(\frac{p+K}{p-2}\right)^\sigma$ is bounded and $\sum_{p>2}(p^{2\sigma}-p^\sigma)^{-1}$ is absolutely convergent for $\mathrm{Re}\, s>1/2$, which proves the lemma.

P r o o f o f t h e o r e m 2. If $\bar{f}(n)=\sum_{f(m)=n}1$ then

$$\sum_{n=1}^{\infty}\bar{f}(n)n^{-s}=\sum_{m=1}^{\infty}(f(m))^{-s}=H(s)$$

and by lemma 1 we have

$$H(s)=A(s)B(s) \tag{10}$$

where

$$A(s)=\sum_{n=1}^{\infty}a(n)n^{-s}=\prod_{p>2}(1-(p+a_{1,1})^{-s})^{-1},$$

$$B(s)=(1+(f(2))^{-s}+(f(2^2))^{-s}+\cdots)\prod_{p>2}(1+a(p,s))$$

so that $B(s)$ is absolutely convergent for $\mathrm{Re}\, s>1/2$. From (10) we obtain

$$N(x)=\sum_{n\le x}\bar{f}(n)=\sum_{n\le x}\sum_{d\mid n}a(d)\,b(n/d)=\sum_{n\le x}b(n)\sum_{m\le x/n}a(m).$$

To estimate $\sum_{m\le y} a(m)$ we need the following theorem due to H. Diamond, [6], on the so-called generalized integers:

Suppose π_n is a non-decreasing sequence tending to ∞ and $\pi_1>1$; then

$$\prod_{n=1}^{\infty}(1-\pi_n^{-s})^{-1}=\prod_{n=1}^{\infty}(1+\pi_n^{-s}+\pi_n^{-2s}+\cdots)=\sum_{i=1}^{\infty}\beta_i\gamma_i^{-s}$$

where $\gamma_1=1, \gamma_2, \gamma_3, \ldots$ is an increasing sequence of positive numbers containing distinct elements of the multiplicative semigroup generated by $\pi_1, \pi_2, \ldots$ and where $\beta_1=1, \beta_2, \beta_3, \ldots$ are non-negative integers. If

$$\sum_{\pi_i\le x}1=\int_2^x \log^{-1}t\cdot dt+O\,(x\exp(-b\log^a x))$$

where $0<a<1$ and $b>0$, then

$$\sum_{\gamma_i\le x}\beta_i=Bx+O\,(x\exp(-c\log^{a/(a+1)}x)$$

for every $c>0$ and $B=\lim_{s\to 1+0}(s-1)\prod_{n=1}^{\infty}(1-\pi_n^{-s})^{-1}$.

If we take $\pi_n=p_n+a_{1,1}$, then since $a_{1,1}$ is an integer $\gamma_i=i$, $\beta_i=a(i)$, and by the prime number theorem (see Walfisz, [8]) for every $\varepsilon_1>0$ and some $b>0$

$$\sum_{\pi_n\le x}1=\sum_{p>2,\,p+a_{1,1}\le x}1=\int_2^x\log^{-1}t\cdot dt+O\,(x\exp(-b\log^{3/5-\varepsilon_1}x)).$$

Diamond's theorem gives then

$$\sum_{n\le x}a(n)=Bx+O\,(x\,\delta(x)) \tag{10}$$

where $\delta(x)=\exp(-c\log^{3/8-\varepsilon}x)$, c and ε are arbitrary positive numbers,

$$B=\lim_{s\to 1+0}(s-1)\prod_{p>2}(1-(p+a_{1,1})^{-s})^{-1}=\frac{1}{2}\prod_{p>2}\left(1-\frac{a_{1,1}}{p\,(p+a_{1,1}-1)}\right). \tag{12}$$

$$N(x)=\sum_{n\le x}b(n)\sum_{m\le x/n}a(m)=Bx\sum_{n\le x}b(n)/n+O(x\sum_{n\le x}|b(n)|\,n^{-1}\,\delta(x/n))=$$

$$x\cdot B\sum_{n=1}^{\infty}b(n)/n+O\,(x\sum_{n>x}|b(n)|\,n^{-1})+O\,(x\sum_{n\le\sqrt{x}}|b(n)|\,n^{-1}\,\delta(x/n))$$

$$+O(\sum_{n>\sqrt{x}}|b(n)|\,n^{-1})=x\cdot B\sum_{n=1}^{\infty}b(n)/n+O\left(x\,\delta\left(\sqrt{x}\right)\right)+O\,(x\cdot x^{-1/4+\varepsilon/2})=$$

$$x\cdot B\sum_{n=1}^{\infty}b(n)/n+O\,(x\exp(-c'\log^{3/8-\varepsilon}x))$$

with perhaps a different constant $c'>0$, since $\sum_{n=1}^{\infty}|b(n)|n^{-1}$ is convergent and $\delta(x)$ is eventually decreasing.

Since $\lim_{s\to 1+0}(s-1)H(s)=\lim_{s\to 1+0}(s-1)A(s)B(1)=$

$$\frac{1}{2}\prod_{p>2}\left(1-\frac{a_{1,1}}{p(p+a_{1,1}-1)}\right)\cdot\sum_{n=1}^{\infty}b(n)n^{-1},$$

the theorem is proved. Theorem 2 may be applied to all the functions mentioned at the beginning of this paper; the constant $C=\lim_{s\to 1+0}(s-1)H(s)$ is easily computed for each of these functions using their defining properties and $\lim_{s\to 1+0}(s-1)\zeta(s)=1$.

P r o o f o f t h e o r e m 3. Since $f(n)$ is multiplicative, $\log f(n)$ is an additive arithmetical function. Asymptotic formulas for sums of reciprocals of additive functions were studied by De Koninck in [4] and De Koninck and Galambos in [5], where a sharper estimate than the one given by theorem 3 is obtained for $f(n)=\sigma(n)$. The method used in [5] is generalized by a forthcoming paper of E. Brinitzer, [2]. The proof of theorem 3 is a direct consequence of theorem 1. Using the fact that $\sum_{2\le n\le x}1/\log n=x/\log x+0(x/\log^2 x)$ and that $f(n)=1$ possibly for $n=2^k$, so that there are $\mathrm{O}(\log x)$ numbers $\le x$ for which $f(n)=1$, we have by (4)

$$\sum_{n\le x}{}'1/\log f(n)\ge\sum_{n\le x}{}'1/(\log n+\log C_1+\log\log\log n)\ge$$
(13)
$$\sum_{n\le x}{}'1/\log n+\mathrm{O}\left(\frac{x\log\log\log x}{\log^2 x}\right)=\frac{x}{\log x}+\mathrm{O}\left(\frac{x\log\log\log x}{\log^2 x}\right).$$

This gives the necessary lower-bound inequality. To prove the upper-bound inequality let from now on m denote an odd number greater than unity, and since $1/\log m-1/(\log C_2+\log m-\log\log\log m)=\mathrm{O}(\log\log\log m/\log^2 m)$ we have by (5)

$$\sum_{n\le x}{}'1/\log f(n)\le\sum_{n\le x}{}'1/\log f(m)\le\sum_{2^k m\le x}1/\log m+\mathrm{O}\left(\frac{x\log\log\log x}{\log^2 x}\right).$$

Using the fact that $\sum_{2\le n\le x}1/\log n=\sum_{2^k m\le x}1/\log 2^k m+\mathrm{O}(\log x)$ and that by partial summation we ob'ain

$$\sum_{m\le x}1/\log m=x/2\log x+\mathrm{O}(x/\log^2 x);\quad\sum_{m\le x}1/\log^2 m=x/2\log^2 x+\mathrm{O}(x/\log^3 x)$$

it follows that

$$\sum_{2^k m\le x}(1/\log m-1/\log 2^k m)\le\sum_{2^k m\le x}k\log 2/\log^2 m=\mathrm{O}\left(\sum_{2^k\le x}\ \sum_{m\le x/2^k}k/\log^2 m\right)=$$
$$\mathrm{O}\sum_{2^k\le x}xk/2^k\log^2(x/2^k+1)=\mathrm{O}(x/\log^2 x)$$

so that finally we obtain

$$\sum_{n\le x}{}' 1/\log f(n) \le \frac{x}{\log x} + O\left(\frac{x\log\log\log x}{\log^2 x}\right) \tag{14}$$

which combined with (13) proves the theorem.

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