PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 22 (36) 1977, pp. 5—22

O-REGULARLY VARYING FUNCTIONS

S. Aljančić, D. Aranđelović

(Communicated June 24, 1977)

1. Introduction and results

A positive, finite and measurable function R, defined on $I_a = [a, \infty[$ for some $a>0$, is said to be regularly varying at infinity if the limit

$$\lim_{x\to\infty} \frac{R(tx)}{R(x)} = r(t) \tag{1.1}$$

is positive and finite for each $t>0$.

(i) (The Characterisation Theorem) If R is a regularly varying (RV) function, then the limit $r(t)$ in (1.1) is necessarily of the form t^ρ for some $-\infty<\rho<\infty$ and for each $t>0$.

The number ρ is the index of R. RV functions of index 0 are called slowly varying (SV) functions and are denoted by L. Their interest lies in the fact that R is a RV function of index ρ if and only if $R(x)=x^\rho L(x)$ on some I_b.

A RV function R of index ρ has the following properties:

(ii) (The Uniform Convergence Theorem) The relation (1.1) holds uniformly for t in any compact interval $I\subset]0, \infty[$.

(iii) The function $\log R$ is locally bounded on I_b for some $b>0$.

(iv) $\lim\limits_{x\to\infty} \dfrac{\log R(x)}{\log x} = \rho$.

(v) $\lim\limits_{x\to\infty} x^{-\sigma} R(x) = \infty$ for $\sigma<\rho$ and $\lim\limits_{x\to\infty} x^{-\tau} R(x)=0$ for $\tau>\rho$.

(vi) (The Representation Theorem) There exists a number $b>0$ such that for $x\geqslant b$

$$R(x) = \exp\left\{\alpha(x) + \int_b^x \beta(t)\frac{dt}{t}\right\}, \tag{1.2}$$

where α and β are bounded measurable functions on I_b such that $\alpha(x)$ converges to a real number and $\beta(x) \to \rho$ as $x \to \infty$. Moreover, without changing the characteristics of α, there exists a function β with a continuous derivative of any specified order such that (1.2) holds.

(vii) For each pair of real numbers σ and τ, $\sigma < \rho < \tau$,[1]

$$\inf_{t \geq x} \{t^{-\sigma} R(t)\} \sim x^{-\sigma} R(x), \qquad \sup_{t \geq x} \{t^{-\tau} R(t)\} \sim x^{-\tau} R(x) \qquad (x \to \infty).$$

(viii) For each pair of real numbers σ and τ, $\sigma < \rho < \tau$,

$$\sup_{b \leq t \leq x} \{t^{-\sigma} R(t)\} \sim x^{-\sigma} R(x), \qquad \inf_{b \leq t \leq x} \{t^{-\tau} R(t)\} \sim x^{-\tau} R(x) \qquad (x \to \infty).$$

(ix) For each $\sigma < \rho$

$$\lim_{x \to \infty} \frac{1}{x^{-\sigma} R(x)} \int_b^x t^{-\sigma} R(t) \frac{dt}{t} = \frac{1}{\rho - \sigma}.$$

(x) For each $\tau > \rho$

$$\lim_{x \to \infty} \frac{1}{x^{-\tau} R(x)} \int_x^\infty t^{-\tau} R(t) \frac{dt}{t} = \frac{1}{\tau - \rho}.$$

Conversely, if the positive, finite and measurable function R on I_a $(a > 0)$ satisfies one of the relations (vi), (vii) or (viii), then R is a *RV* function of index ρ. The same is true if for some real σ (τ, respectively) the limit in (ix) (in (x)) exists and is positive and finite, the index ρ of the *RV* function R being then determined by the equation in (ix) (in (x)).

RV functions have been introduced by J. Karamata [1, 2]. He proved for continuous functions R the crucial of the here mentioned results. Since then, many authors have contributed to the development of the theory of *RV* functions and their application in different fields. In this respect we refer to the recent book of E. Seneta [3].

Parallel to *RV* functions one can consider *O-RV* functions.

A positive, finite and measurable function K on I_a $(a > 0)$ is said to be O-regularly varying at infinity if

$$\limsup_{x \to \infty} \frac{K(tx)}{K(x)} = r(t) \tag{1.3}$$

is finite for each $t > 0$.[2]

---

[1] $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ $(x \to \infty)$.

[2] Or, equivalently: *if*

$$\liminf_{x \to \infty} \frac{K(tx)}{K(x)} > 0 \qquad \textit{and} \qquad \limsup_{x \to \infty} \frac{K(tx)}{K(x)} < \infty$$

for each $t \geq 1$.

From

$$\frac{K(stx)}{K(x)}=\frac{K(stx)}{K(tx)}\frac{K(tx)}{K(x)}$$

there follows, as $x\to\infty$,

(1.4) $$r(st)\leqslant r(s)\,r(t) \quad \text{for each positive } s \text{ and } t.$$

If $r(t)=0$ for some $t>0$, (1.4) would imply $1=r(1)\leqslant r(t)\,r(1/t)=0$. *Hence, $r(t)>0$ for each $t>0$.* So,

$$\liminf_{x\to\infty}\frac{K(tx)}{K(x)}=\frac{1}{r(1/t)}$$

is positive and finite for each $t>0$ too. Consequently, *K is a O-RV function at infinity if and only if*

(1.5) $$K(tx)\asymp K(x)\ (x\to\infty) \quad \text{for each } t>0.$$[1]

In this form[2] *O-RV* functions were introduced by V. G. Avakumović [4] in a note concerning some tauberian theorems, but it was J. Karamata [5] who in 1936 revealed their characteristic properties. It happened that N. K. Bari and S. B. Stečkin [6] in their well-known memoir on best approximation, which appeared in 1956, independently introduced monotone *O-RV* functions which tend to zero and developed their theory.

J. Karamata proved that a *O-RV* function K can be characterized by each of the following conditions[3]:

(K_I) There exist measurable and bounded real functions α and β on I_b for some $b\geqslant a$ such that for $x\geqslant b$

(1.6) $$K(x)=\exp\left\{\alpha(x)+\int_b^x\beta(t)\frac{dt}{t}\right\}.$$

(K_{II}) There exist four numbers $0<m<M<\infty$ and $\sigma<\tau$ such that for $y\geqslant x\geqslant b$[4]

(1.7) $$m\left(\frac{y}{x}\right)^{\sigma}\leqslant\frac{K(y)}{K(x)}\leqslant M\left(\frac{y}{x}\right)^{\tau}.$$

---

[1] $f(x)\asymp g(x)$ $(x\to\infty)$ means that there exists two numbers $0<m<M<\infty$ such that (*) $m\leqslant f(x)/g(x)\leqslant M$ holds for x large enough. If we wish to precise that (*) holds for $x\geqslant a$, we write $f(x)\asymp g(x)$ on I_a.

[2] In fact, J. Karamata supposed that for a given $\lambda\in]1,\infty[$ there exist $m=m(\lambda)$ and $M=M(\lambda)$ such that

$$0<m(\lambda)\leqslant\frac{K(tx)}{K(x)}\leqslant M(\lambda)<\infty \quad \text{for each } t\in[1,\lambda].$$

E. Seneta [3] supposes a priori that m and M are independent of λ.

[3] Except (K'_{III}) which occurs in E. Seneta [3]. In proving our Theorem 3 we prove incidentally the equivalence of Karamata's conditions (K_I), (K_{II}), (K_{III}) and (K'_{III}); for the sake of completeness their equivalence to (1.3) is proved in 2.7.

[4] Another way to express (1.7) is to say that the function $x^{-\sigma}K(x)$ almost increases and that the function $x^{-\tau}K(x)$ almost decreases on I_b (see 2.1).

(K_{III}) There exists a real number σ such that, as $x \to \infty$,

$$\int_b^x t^{-\sigma} K(t) \frac{dt}{t} \asymp x^{-\sigma} K(x) \text{ on } I_c \text{ for each } c > b. \tag{1.8}$$

(K'_{III}) There exists a real number τ such that, as $x \to \infty$,

$$\int_x^\infty t^{-\tau} K(t) \frac{dt}{t} \asymp x^{-\tau} K(x) \text{ on } I_b. \tag{1.9}$$

Remark. Moreover, if (1.6) holds, then there exist measurable and bounded real functions $\overline{\alpha}$ and $\overline{\beta}$ on I_b, the function $\overline{\beta}$ being *continuous*, such that

$$K(x) = \exp\left\{\overline{\alpha}(x) + \int_b^x \overline{\beta}(t) \frac{dt}{t}\right\} \tag{$\widehat{1.6}$}$$

holds. Indeed, one can take for

$$\overline{\beta}(x) = \int_x^{ex} \beta(t) \frac{dt}{t} = \int_1^e \beta(xt) \frac{dt}{t} \tag{1.10}$$

and

$$\overline{\alpha}(x) = \alpha(x) + \int_b^x \{\beta(t) - \overline{\beta}(t)\} \frac{dt}{t} \tag{1.11}$$

$$= \alpha(x) + \int_1^e \{\beta(bt) - \beta(xt)\}(1 - \log t) \frac{dt}{t}.$$

We note that $\overline{\beta}$ earns some properties of β: if β is monotone or convex, so is $\overline{\beta}$. Also:

$$\beta(t) \geqslant m \text{ for almost all } t \geqslant c \text{ implies } \overline{\beta}(t) \geqslant m \text{ for } t \geqslant c,$$

$$\beta(t) \leqslant M \text{ for almost all } t \geqslant c \text{ implies } \overline{\beta}(t) \leqslant M \text{ for } t \geqslant c.$$

In particular, we have for $x \geqslant b$

$$\inf_{t \geqslant x} \overline{\beta}(t) \geqslant \operatorname*{ess\,inf}_{t \geqslant x} \beta(t) \text{ and } \sup_{t \geqslant x} \overline{\beta}(t) \leqslant \operatorname*{ess\,sup}_{t \geqslant x} \beta(t)$$

$$\liminf_{x \to \infty} \overline{\beta}(x) \geqslant \liminf_{x \to \infty} \beta(t) \text{ and } \limsup_{x \to \infty} \overline{\beta}(x) \leqslant \limsup_{x \to \infty} \beta(x).$$

Repeating this procedure ($\alpha_{-1}=\alpha$, $\beta_{-1}=\beta$, $\alpha_n=\overline{\alpha}_{n-1}$, $\beta_n=\overline{\beta}_{n-1}$ for $n=1, 2, \ldots$), the function K given by (1.6) can be written, for any natural number N, in the form

$$K(x)=\exp\left\{\alpha_N(x)+\int_b^x \beta_N(t)\,\frac{dt}{t}\right\} \quad (x\geqslant b), \tag{1.6$_N$}$$

where α_N is a measurable and bounded and β_N a bounded and N times continuosly differentiable real function on I_b.

Theorem 1. *If K is O-RV function and I any compact interval in* $]0, \infty[$, *then*

$$\limsup_{x\to\infty}\frac{K(tx)}{K(x)}<\infty$$

holds uniformly for t in I, i.e.

$$\limsup_{x\to\infty}\sup_{t\in I}\frac{K(tx)}{K(x)}<\infty.$$

Corollary 1. *If I is any compact interval in* $]0, \infty[$, *then*

$$\liminf_{x\to\infty}\frac{K(tx)}{K(x)}>0 \quad \textit{uniformly for } t \textit{ in } I.$$

Corollary 2. *The function* $\log r$ *defined by* (1.3) *is bounded on any compact interval in* $]0, \infty[$.

Corollary 3. *There exists a number $b\geqslant a$ such that the function* $\log K$ *is locally bounded on I_b.*

Indeed, let m, M and b be positive real numbers such that

$$m\,K(x)\leqslant K(tx)\leqslant MK(x) \text{ for each } t\in[1, e] \text{ and each } x\geqslant b.$$

Then, any $s\in[be^n, be^{n+1}]$ $(n=0, 1, 2, \ldots)$ is of the form $s=tbe^n$ with $t\in[1, e]$, so that

$$m\,K(be^n)\leqslant K(tbe^n)=K(s)\leqslant MK(be^n).$$

Theorem 2. *Let K be a O-RV function and let r be the positive and finite function on* $]0, \infty[$ *defined by* (1.3). *Then*

1° *the limits*

$$p=p(K)=\lim_{t\to 0+}\frac{\log r(t)}{\log t} \quad \textit{and} \quad q=q(K)=\lim_{t\to\infty}\frac{\log r(t)}{\log t} \quad \textit{exist}; \tag{1.12}$$

2° $-\infty<p\leqslant q<\infty$;

3° $r(t)\geqslant\max\{t^p, t^q\}$ *for each* $t>0$;

4° *For any pair of real numbers p' and q', $p'<p$ and $q'>q$, there exist a real number $M\geqslant 1$ such that*

$$r(t)\leqslant M \max\{t^{p'}, t^{q'}\} \quad \textit{for each } t>0.$$

The numbers p and q are the *lower* and the *upper index* of the *O-RV* function K. If $p=q$, we say that K is of *index* p. Of special interest are *O-RV* functions of index $p=0$; we call them "slow" *O-RV* functions.

For a given *O-RV* function K define:

$$\underline{\rho}=\underline{\rho}(K)=\sup_{\beta}\liminf_{x\to\infty}\beta(x), \tag{1.13}$$

$$\overline{\rho}=\overline{\rho}(K)=\inf_{\beta}\limsup_{x\to\infty}\beta(x), \tag{1.13'}$$

where the sup and inf are taken over all measurable and bounded functions β on I_b for which there exist a measurable and bounded function α on I_b such that (1.6) holds:[1])

$$\underline{\rho}_1=\underline{\rho}_1(K)=\sup\{\sigma\in\mathbf{R}\mid x^{-\sigma}K(x) \text{ almost increases on } I_b\}, \tag{1.14}$$

$$\overline{\rho}_1=\overline{\rho}_1(K)=\inf\{\tau\in\mathbf{R}\mid x^{-\tau}K(x) \text{ almost decreases on } I_b\}; \tag{1.14'}$$

$$\underline{\rho}_2=\underline{\rho}_2(K)=\sup\left\{\sigma\in\mathbf{R}\;\middle|\;\int_b^x t^{-\sigma}K(t)\frac{dt}{t}\asymp x^{-\sigma}K(x) \text{ on } I_c \text{ for each } c>b\right\}, \tag{1.15}$$

$$\overline{\rho}_2=\overline{\rho}_2(K)=\inf\left\{\tau\in\mathbf{R}\;\middle|\;\int_x^\infty t^{-\tau}K(t)\frac{dt}{t}\asymp x^{-\tau}K(x) \text{ on } I_b\right\}. \tag{1.15'}$$

Suppose, in the sequel, that K is a *O-RV* function of lower and upper index p and q, respectively.

Theorem 3.

$$p=\underline{\rho}=\underline{\rho}_1=\underline{\rho}_2; \qquad q=\overline{\rho}=\overline{\rho}_1=\overline{\rho}_2. \tag{1.16}$$

Theorem 4.

$$\lim_{x\to\infty} x^{-\sigma}K(x)=\infty \quad \textit{for each } \sigma<p, \tag{1.17}$$

$$\lim_{x\to\infty} x^{-\tau}K(x)=0 \quad \textit{for each } \tau>q. \tag{1.17'}$$

---

[1]) We note that the same numbers $\underline{\rho}$ and $\overline{\rho}$ are obtained if we take in (1.13) and (1.13)' respectively the sup and inf over all functions β which are N times continously differentiable on I_b and that $\overline{\rho}(K)=-\underline{\rho}(1/K)$.

Theorem 5. *If* $\sigma<p$ *the following equivalent relations are true:*

(1.18) $$\inf_{t\geqslant x} t^{-\sigma}K(t)\asymp x^{-\sigma}K(x) \text{ on } I_b,$$

(1.19) $$\sup_{b\leqslant t\leqslant x} t^{-\sigma}K(t)\asymp x^{-\sigma}K(x) \text{ on } I_b,$$

(1.20) *there exists a positive nondecreasing function* φ *on* I_b *such that* $K(x)\asymp x^{\sigma}\varphi(x)$ *on* I_b,

If $\tau>q$ *the following equivalent relations are true:*

(1.18′) $$\sup_{t\geqslant x} t^{-\tau}K(t)\asymp x^{-\tau}K(x) \text{ on } I_b,$$

(1.19′) $$\inf_{b\leqslant t\leqslant x} t^{-\tau}K(t)\asymp x^{-\tau}K(x) \text{ on } I_b,$$

(1.20′) *there exists a positive nonincreasing function* ψ *on* I_b *such that* $K(x)\asymp x^{\tau}\psi(x)$ *on* I_b.

2. Proof of theorems

2.1. *Almost monotone functions.* A function f positive and finite on I_b is said to be *almost increasing on* $I_c (c\geqslant b)$, in symbols $f(x)\nearrow$ for $x\geqslant c$, if there exists a constant $M\geqslant 1$ such that

(2.1) $$f(x)\leqslant Mf(y) \text{ for each } y\geqslant x\geqslant c,$$

or, equivalently, if

(2.2) $$f(x)\leqslant M\inf_{t\geqslant x} f(t) \text{ for each } x\geqslant c.$$

A function $f(x)$ is said to be *almost increasing when* $x\to\infty$ (i.e. for x large enough) if it is almost increasing on some interval I_c, or, equivalently, if

(2.3) $$f(x)\preccurlyeq \inf_{t\geqslant x} f(t) \qquad (x\to\infty).$$

We note that the last relation has the same meaning as

(2.4) $$f(x)\asymp \inf_{t\geqslant x} f(t) \qquad (x\to\infty).$$

By duality (with respect to the ordered set of positive real numbers in which f takes his values), a function f is said to be *almost decreasing on* I_c, in symbols $f(x)\searrow$ for $x\geqslant c$, if there exists a constant $0<m\leqslant 1$ such that

(2.1′) $$f(x)\geqslant mf(y) \text{ for each } y\geqslant x\geqslant c,$$

or, equivalently,

(2.2′) $$f(x)\geqslant m\sup_{t\geqslant x} f(t) \text{ for each } x\geqslant c.$$

Similarly, a function $f(x)$ is said to be *almost decreasing when* $x\to\infty$ if

$$(2.3') \qquad f(x) \succcurlyeq \sup_{t\geqslant x} f(t) \qquad (x\to\infty),$$

or, equivalently, if

$$(2.4') \qquad f(x) \asymp \sup_{t\geqslant x} f(t) \qquad (x\to\infty).$$

Almost increasing and almost decreasing functions are said to be almost monotone functions. Since $f(x)$ is almost increasing for $x\geqslant c$ if and only if $1/f(x)$ is almost decreasing for $x\geqslant c$, it is sufficient to consider only almost increasing functions. We list here some properties of such functions.

(i) If f is almost increasing on I_c, then f is there *bounded away from* 0 $(\inf_{t\geqslant c} f(t)>0)$ and *locally bounded from above* $(\sup_{c\leqslant t\leqslant x} f(t)<\infty$ for each $x\geqslant c)$.

(ii) If f is almost increasing on I_c, then f is bounded from above on I_c *or* $f(\infty)=\infty$. [From (2.2) follows $\limsup_{x\to\infty} f(x)\leqslant M \liminf_{x\to\infty} f(x)$].

(iii) The function f is almost increasing *and* almost decreasing on I_c if and only if f is bounded away from both 0 and ∞ $(0<m\leqslant f(x)\leqslant M<\infty)$.

(iv) The product of two almost increasing functions on I_c is an almost increasing function (on I_c). If f is an almost increasing function on I_c, such are f^α and $x^\alpha f(x)$ for each $\alpha>0$.

(v) If $f(x)$ is almost increasing when $x\to\infty$ and if $g(x)\to\infty (x\to\infty)$, then $\lim_{x\to\infty} g(x)f(x)=\infty$; in particular, $\lim_{x\to\infty} x^\alpha f(x)=\infty$ for each $\alpha>0$.

(vi) A function f is almost increasing on I_c if and only if there exits an increasing function φ on I_c such that $f(x)\asymp\varphi(x)$ on I_c. [Sufficiency: if $m\varphi\leqslant \leqslant f\leqslant M\varphi$ on I_c, then $f(x)\leqslant M\varphi(x)\leqslant M\varphi(y)\leqslant m^{-1}Mf(y)$ for each $y\geqslant x\geqslant c$. Necessity follows from (2.4): $\varphi(x)=\inf_{t\geqslant x} f(t)$].

(vii) If f is almost increasing on an interval, it is such on each subinterval. If f is almost increasing on two intervals with nonvoid intersection, then it is such on their union.

(viii) If $\log f$ is locally bounded on I_c (bounded on $[c, d]$ for each $d\geqslant c$), then the following relations are equivalent:

$$(2.1) \qquad f \text{ is almost increasing on } I_c;$$

$$(2.5) \qquad \sup_{c\leqslant t\leqslant x} f(t) \preccurlyeq f(x)\ (x\to\infty);$$

$$(2.6) \qquad \sup_{c\leqslant t\leqslant x} f(t) \asymp f(x)\ (x\to\infty).$$

[The relation (2.1) is equivalent to

$$(2.7) \qquad \sup_{c\leqslant t\leqslant x} f(t)\leqslant Mf(x) \text{ for each } x\geqslant c.$$

If f is almost increasing on I_c, then (2.5) follows on account of (2.7). If (2.5) holds, then (2.7) follows with some $d \geqslant c$ instead of c, and so f is almost increasing on I_d. Being bounded away from both 0 and ∞ on $[c, d]$, f is almost increasing on $[c, d]$ and, consequently, on I_c too.]

(ix) If f is *measurable* and almost increasing on I_c, then

(2.8) $$F_\alpha(x) \stackrel{\text{def}}{=} \int_c^x t^{-\alpha} f(t) \frac{dt}{t} \preccurlyeq x^{-\alpha} f(x) \quad \text{on } I_c \text{ for each } \alpha<0,$$

(2.9) $$F^*_\alpha(x) \stackrel{\text{def}}{=} \int_x^\infty t^{-\alpha} f(t) \frac{dt}{t} \succcurlyeq x^{-\alpha} f(x) \quad \text{on } I_c \text{ for each } \alpha>0,$$

for which $F^*_\alpha(c)$ is finite.

[Indeed, from $f(t) \leqslant Mf(x)$ for $c \leqslant t \leqslant x$, there follows

$$F_\alpha(x) \leqslant Mf(x) \int_c^x t^{-\alpha} \frac{dt}{t} \leqslant Mf(x) \int_0^x x^{-\alpha} \frac{dt}{t} = -\frac{M}{\alpha} x^{-\alpha} f(x),$$

and, similarly, $Mf(t) \geqslant f(x)$ for $t \geqslant x \geqslant c$ implies

$$MF^*_\alpha(x) \geqslant f(x) \int_x^\infty t^{-\alpha} \frac{dt}{t} = \frac{1}{\alpha} x^{-\alpha} f(x).]$$

(ix′) If f is *measurable* and almost decreasing on I_c, then

(2.8′) $$F^*_\alpha(x) \preccurlyeq x^{-\alpha} f(x) \text{ on } I_c \text{ for each } \alpha>0,$$

(2.9′) $$F_\alpha(x) \succcurlyeq x^{-\alpha} f(x) \text{ on } I_{c'} \text{ for each } \alpha<0 \text{ and each } c'>c.$$

[The proof is similar. From $mf(t) \leqslant f(x)$ for $t \geqslant x \geqslant c$ there follows

$$m F^*_\alpha(x) = m \int_x^\infty t^{-\alpha} f(t) \frac{dt}{t} \leqslant f(x) \int_x^\infty t^{-\alpha} \frac{dt}{t} = \frac{1}{\alpha} x^{-\alpha} f(x) \text{ for } x \geqslant c$$

and $f(t) \geqslant mf(x)$ for $x \geqslant t \geqslant c$ implies

$$F_\alpha(x) = \int_c^x t^{-\alpha} f(t) \frac{dt}{t} \geqslant mf(x) \int_c^x t^{-\alpha} \frac{dt}{t} \geqslant mf(x) \int_{x\frac{c}{c'}}^x t^{-\alpha} \frac{dt}{t}$$

$$\geqslant \frac{m}{-\alpha}\left(1 - \left(\frac{c}{c'}\right)^{-\alpha}\right) x^{-\alpha} f(x) \text{ for } x \geqslant c'.]$$

2.2. *Proof of Theorem* 1. The proof follows the same ideas as the proof of Theorem 2.12 in E. Seneta [3]. We give it here for the sake of completeness.

The Theorem holds for $I=[1, e]$. For $n=1, 2, \dots$ let

$$d_n=\sup_{x\geqslant n}\sup_{t\in I}\frac{K(tx)}{K(x)}, \quad 0<c_n<d_n \quad \lim c_n=\lim d_n$$

and suppose that $x_n\geqslant n$ and $t_n\in I$ are such that

$$c_n<\frac{K(t_n x_n)}{K(x_n)} \quad (\leqslant d_n).$$

It is enough to prove that the sequence $(K(t_n x_n)/K(x_n))$ is bounded.

The sets

$$U_{mn}=\bigcap_{j\geqslant n}\{t\in[1, e^2]\,|\,K(tx_j)/K(x_j)\leqslant m\},$$

$$V_{mn}=\bigcap_{j\geqslant n}\{t\in[1, e^2]\,|\,K(t_j x_j)/K(t\,t_j x_j)\leqslant m\}$$

are measurable and increasing with respect to m and n; they tend to the interval $[1, e^2]$ when m and n tend to infinity. Choose the natural numbers M and N such that

$$\int_{U_{MN}}\frac{dt}{t}>\frac{3}{2} \qquad \int_{V_{MN}}\frac{dt}{t}>\frac{3}{2}.$$

It follows then, for $n\geqslant N$,

$$\int_{U_{Mn}}\frac{dt}{t}>\frac{3}{2} \qquad \int_{t_n V_{Mn}}\frac{dt}{t}=\int_{V_{Mn}}\frac{dt}{t}>\frac{3}{2}.$$

Since the sets U_{Mn} and $t_n V_{Mn}$ are in $[1, e^3]$, their intersection is nonvoid; let $s_n\in U_{Mn}\cap t_n V_{Mn}$. Since $s_n\in U_{Mn}$ and $s_n/t_n\in V_{Mn}$, it follows

$$\frac{K(t_n x_n)}{K(x_n)}=\frac{K(t_n x_n)}{K((s_n/t_n)\,t_n x_n)}\,\frac{K(s_n x_n)}{K(x_n)}\leqslant M^2 \text{ for } n\geqslant N.$$

The Theorem holds for $I=[e^n, e^{n+1}]=e^n[1, e]$ *for each integer* n. Let b and M be positive real numbers such that

$$\frac{K(tx)}{K(x)}\leqslant M \qquad \text{for each } t\in[1, e] \text{ and each } x\geqslant b.$$

For $s\in e^n[1, e]$ we have $t=e^{-n}s\in[1, e]$ and $y=tx\geqslant x$, so that

$$\frac{K(sx)}{K(x)}=\frac{K(e^n y)}{K(y)}\,\frac{K(tx)}{K(x)}\leqslant M\,\frac{K(e^n y)}{K(y)} \quad \text{for each } s\in I \text{ and each } x\geqslant b.$$

The statement then follows from the finiteness of $r(e^n)$.

The Theorem holds for any compact interval in $]0, \infty[$ since such an interval can be covered by a finite number of intervals $[e^n, e^{n+1}]$.

2.3. *Proof of Theorem* 2. The proof is based on the following lemma 1 and its corollaries.

Lemma 1. *Let the function* ρ *be positive, finite and locally bounded from above on* $]1, \infty[$ *and such that* $\rho(st) \leqslant \rho(s)\rho(t)$ *for each* s *and* $t > 1$. *Then*

$$\lim_{t\to\infty} \frac{\log \rho(t)}{\log t} = \inf_{t>1} \frac{\log \rho(t)}{\log t}. \tag{2.10}$$

Proof. It is enough to prove that

$$\limsup_{t\to\infty} \frac{\log \rho(t)}{\log t} \leqslant \frac{\log \rho(t)}{\log t} \quad \text{for each } t > 1.$$

Fix a $t > 1$ and let $\rho(s) \leqslant M$ for $1 \leqslant s \leqslant t$. For $n = 1, 2, \dots$ and $1 \leqslant s \leqslant t$ we have first

$$\rho(s t^n) \leqslant \rho(s)\, \rho(t)^n \leqslant M \rho(t)^n,$$

so that

$$\log \rho(st^n) \leqslant \log M + n \log \rho(t).$$

Dividing by $\log(st^n) > 0$, we obtain

$$\frac{\log \rho(st^n)}{\log(st^n)} \leqslant \frac{\log M + n \log \rho(t)}{\log s + n \log t} \leqslant \frac{\log M + n \log \rho(t)}{c_n + n \log t},$$

where $0 \leqslant c_n \leqslant \log t$. For $x \geqslant t^k$ $(k = 1, 2, \dots)$, there exists some integer $m \geqslant k$ such that $t^m \leqslant x < t^{m+1}$. By the preceding inequality

$$\sup_{x \geqslant t^k} \frac{\log \rho(x)}{\log x} = \sup_{k \leqslant n} \sup_{t^n \leqslant u < t^{n+1}} \frac{\log \rho(u)}{\log u} \leqslant \sup_{n \geqslant k} \frac{\log M + n \log \rho(t)}{c_n + n \log t},$$

and the required statement follows by letting $k \to \infty$.

Corollary 1. *Let the function* ρ *be positive, finite and locally bounded from above on* $]0, 1]$ *and such that* $\rho(st) \leqslant \rho(s)\rho(t)$ *for each* s *and* t *in* $]0, 1]$. *Then*

$$\lim_{t\to 0+} \frac{\log \rho(t)}{\log t} = \sup_{0<t<1} \frac{\log \rho(t)}{\log t}. \tag{2.11}$$

Proof. One has only to apply Lemma 1 to the function $\rho(1/t)$ $(t \geqslant 1)$.

Corollary 2. *Let the function* ρ *be positive, finite and locally bounded from above on* $]0, \infty[$ *and such that* $\rho(st) \leqslant \rho(s)\rho(t)$ *for each* s *and* $t > 0$. *Denote by* q *and* p *the limits in* (2.10) *and* (2.11) *respectively. Then*

$$-\infty < p \leqslant q < \infty. \tag{2.12}$$

Proof. $q < \infty$ and $p > -\infty$ follow from (2.10) and (2.11) respectively. Since $\rho(1) = \rho(1 \cdot 1) \leqslant \rho(1)\rho(1)$, we have $\rho(1) \geqslant 1$. Hence, $1 \leqslant \rho(1) \leqslant \rho(1/t)\rho(t)$ for each $t > 1$, and consequently $-\log \rho(1/t) \leqslant \log \rho(t)$ and so

$$\frac{\log \rho(1/t)}{\log 1/t} \leqslant \frac{\log \rho(t)}{\log t}.$$

The statement $p \leqslant q$ follows by letting $t \to \infty$.

Corollary 3. *Under the hypotheses of Corollary* 2, *we have first*

$$\max\{t^p, t^q\} \leqslant \rho(t) \text{ for each } t > 0. \tag{2.13}$$

On the other side, for each pair of numbers $p' < p$ and $q' > q$, there exists a real number $M \geqslant 1$ such that

$$\rho(t) \leqslant M \max\{t^{p'}, t^{q'}\} \text{ for each } t > 0. \tag{2.14}$$

Proof. From (2.10) and (2.11) follows

$$\frac{\log \rho(t)}{\log t} \begin{cases} \leqslant p & \text{for } 0 < t < 1, \\ \geqslant q & \text{for } t > 1, \end{cases}$$

so that

$$\rho(t) \begin{cases} \geqslant t^p & \text{for } 0 < t < 1, \\ \geqslant t^q & \text{for } t > 1. \end{cases}$$

The statement (2.13) then follows on account of (2.12) and $\rho(1) \geqslant 1$.

For $p' < p$ and $q' > q$, there exist two real numbers a and b, $0 < a < 1 < b$ such that

$$\frac{\log \rho(t)}{\log t} \begin{cases} \geqslant p' & \text{for } 0 < t < a, \\ \leqslant q' & \text{for } t > b, \end{cases}$$

or

$$\rho(t) \begin{cases} \leqslant t^{p'} & \text{for } 0 < t < a, \\ \leqslant t^{q'} & \text{for } t > b. \end{cases}$$

To obtain the inequality in (2.14) it is enough to take for M the supremum of the function $\rho(t)/\max\{t^{p'}, t^{q'}\}$ on the interval $[a, b]$.

The proof of Theorem 2 is now immediate, since the function r of Theorem 2 satisfies all hypotheses required for the function ρ in Lemma 1 and its Corollaries (definition (1.3) and its immediate consequence, Corollary 2 of Theorem 1 and relation (1.4)).

2.4. *Proof of Theorem* 3. We shall prove Theorem 3 by combining the statements of a number of lemmas.

Lemma 2. *Let K be of the form* (1.6). *Then the function $x^{-\lambda} K(x)$ almost increases on I_b for $\lambda < \liminf_{x\to\infty} \beta(x)$ and almost decreases on I_b for $\lambda > \limsup_{x\to\infty} \beta(x)$.*

Remark. Using the numbers defined by (1.13), (1.13'), (1.14) and (1.14'), the result of Lemma 2 may be stated as follows:

$$\underline{\rho}(K) \leqslant \underline{\rho_1}(K) \text{ and } \overline{\rho}(K) \geqslant \overline{\rho_1}(K). \tag{2.15}$$

Proof. From (1.6) follows

$$\frac{y^{-\mu}K(y)}{x^{-\mu}K(x)}=\exp\left\{\alpha(y)-\alpha(x)+\int_x^y(\beta(t)-\mu)\frac{dt}{t}\right\} \tag{2.16}$$

for $y\geqslant x\geqslant b$ and for any real number μ.

The function $\exp\alpha(x)$ being bounded away from both 0 and ∞, and so almost increasing and almost decreasing on I_b, we can assume that $\alpha=0$. For the same reason the function K is almost increasing and almost decreasing on every finite interval $[b, c]$ $(c\geqslant b)$. Hence, it is sufficient to prove that lemma 1 holds on some interval I_c $(c\geqslant b)$. Choose $c\geqslant b$ such that $\beta(t)\geqslant\lambda$ $(\leqslant\lambda)$ for $t\geqslant c$. From (2.16) with $\mu=\lambda$ then follows

$$\frac{y^{-\lambda}K(y)}{x^{-\lambda}K(x)}=\exp\left\{\int_x^y\{\beta(t)-\lambda\}\frac{dt}{t}\right\}\geqslant\exp 0=1\ (\leqslant\exp 0=1).$$

for $y\geqslant x\geqslant c$.

Lemma 3. *Let K be a positive and measurable function on I_b. Suppose that there exist two numbers σ_0 and τ_0 such that $x^{-\sigma_0}K(x)$ almost increases and that $x^{-\tau_0}K(x)$ almost decreases on I_b. Then*

$$\int_b^x t^{-\sigma}K(t)\frac{dt}{t}\asymp x^{-\sigma}K(x)\ \text{on}\ I_c \tag{2.17}$$

for each $c>b$ and each $\sigma<\sigma_0$,

$$\int_x^\infty t^{-\tau}K(t)\frac{dt}{t}\asymp x^{-\tau}K(x)\ \text{on}\ I_b \tag{2.17'}$$

for each $\tau>\tau_0$.

Proof. The Lemma follows immediately from the properties (ix) and (ix') of almost monotone functions: one obtains (2.17) by putting $f(t)=t^{-\sigma_0}K(t)$ and $\alpha=\sigma-\sigma_0$ in (2.8) and (2.9'), and, similarly, (2.17') follows by putting $f(t)=t^{-\tau_0}K(t)$ and $\alpha=\tau-\tau_0$ in (2.9) and (2.8').

Remark. Using the numbers defined by (1.14), (1.14'), (1.15) and (1.15'), the result of Lemma 3 can be stated as follows:

$$\underline{\rho_1}(K)\leqslant\underline{\rho_2}(K)\ \text{and}\ \overline{\rho_2}(K)\leqslant\overline{\rho_1}(K). \tag{2.18}$$

The previous reasoning and lemma 3 show that the sets in (1.15) and (1.15') are intervals.

Lemma 4. *Let K be a positive, measurable and locally bounded function on I_b. Suppose that for a real number σ (2.17) holds for some $c>b$. Then K can* *be written in the form* (1.6), *where* α *and* β *are measurable bounded functions on* I_b *and* $\liminf_{x\to\infty} \beta(x) > \sigma$.

Proof. For $b \leqslant x < c$ we can take $\alpha(x) = \log K(x)$ and $\beta(x) = 0$; hence, it is enough to determine α and β for $x \geqslant c$. Let m and M be real numbers such that $0 < m \leqslant 1 \leqslant M$ and

$$m \leqslant G(x) = \frac{x^{-\sigma} K(x)}{\int\limits_b^x t^{-\sigma} K(t) \frac{dt}{t}} \leqslant M \quad \text{for } x \geqslant c.$$

Since

$$\int\limits_c^y G(x) \frac{dx}{x} = \left[\log \int\limits_b^x t^{-\sigma} K(t) \frac{dt}{t}\right]_c^y \quad (y \geqslant c),$$

and so $(x \geqslant c)$

$$K(x) = x^{\sigma} G(x) \int\limits_b^x t^{-\sigma} K(t) \frac{dt}{t} = \left\{c^{\sigma} G(x) \int\limits_b^c t^{-\sigma} K(t) \frac{dt}{t}\right\} \left\{\left(\frac{x}{c}\right)^{\sigma} \exp \int\limits_c^x G(t) \frac{dt}{t}\right\}.$$

Lemma 4 follows if we put

$$\alpha(x) = \log \left\{c^{\sigma} G(x) \int\limits_b^c t^{-\sigma} K(t) \frac{dt}{t}\right\} \quad \text{and} \quad \beta(x) = G(x) + \sigma.$$

Lemma 4′. *Let* K *be a positive, measurable and locally bounded function on* I_b. *Suppose that for a real number* τ (2.17′) *holds. Then* K *can be written in the form* (1.6), *where* α *and* β *are measurable bounded functions on* I_b *and* $\limsup_{x\to\infty} \beta < \tau$.

Proof. Let m and M be real numbers such that $0 < m \leqslant 1 \leqslant M$ and

$$m \leqslant H(x) = \frac{x^{-\tau} K(x)}{\int\limits_x^{\infty} t^{-\tau} K(t) \frac{dt}{t}} \leqslant M \quad \text{for } x \geqslant b.$$

Since

$$\int\limits_b^y H(t) \frac{dt}{t} = -\left[\log \int\limits_x^{\infty} t^{-\tau} K(t) \frac{dt}{t}\right]_b^y = \log \frac{\int\limits_b^{\infty} t^{-\tau} K(t) \frac{dt}{t}}{\int\limits_y^{\infty} t^{-\tau} K(t) \frac{dt}{t}} \quad (y \geqslant b).$$

and so $(x \geqslant b)$

$$K(x) = x^{\tau} H(x) \int\limits_{x}^{\infty} t^{-\tau} K(t) \frac{dt}{t} =$$

$$= \left\{ b^{\tau} H(x) \int\limits_{b}^{\infty} t^{-\tau} K(t) \frac{dt}{t} \right\} \left\{ \left(\frac{x}{b} \right)^{\tau} \exp \int\limits_{b}^{x} (-H(t)) \frac{dt}{t} \right\}.$$

Lemma 4′ follows if we put

$$\alpha(x) = \log \left\{ b^{\tau} H(x) \int\limits_{b}^{\infty} t^{-\tau} K(t) \frac{dt}{t} \right\} \text{ and } \beta(x) = \tau - H(x).$$

Remark. Using the numbers defined by (1.15), (1.15′), (1.13) and (1.13′) the result of Lemma 4 and of Lemma 4′ can be stated as

(2.19) $$\underline{\rho_2}(K) \leqslant \underline{\rho}(K) \text{ and } \overline{\rho}(K) \leqslant \overline{\rho_2}(K).$$

respectively.

From the Lemmas 2, 3, 4 and 4′ follows the equivalence of Karamata's conditions (K_I), (K_{II}), (K_{III}) and (K'_{IV}) by the scheme

$$\begin{array}{ccccc} & & (K_{III}) & & \\ & \swarrow \text{L. 4} & & \nwarrow \text{L. 3} & \\ (K_I) & & \xrightarrow{\text{L. 2}} & & (K_{II}) \\ & \nwarrow \text{L. 4}' & & \swarrow \text{L. 3} & \\ & & (K'_{III}) & & \end{array}$$

Moreover, the relations (2.15), (2.18) and (2.19) imply

(2.20) $$\underline{\rho}(K) = \underline{\rho_1}(K) = \underline{\rho_2}(K) \text{ and } \overline{\rho}(K) = \overline{\rho_1}(K) = \overline{\rho_2}(K).$$

The proof of Theorem 3 completes

Lemma 5. *Let K be a O-RV function of lower and upper index p and q respectively. Then*

(2.21) $$p = \underline{\rho_1}(K) \text{ and } q = \overline{\rho_1}(K).$$

Proof. We shall prove the second relation in (2.21); by applying it to the function $1/K$, one obtains the first one.

If $x^{-\tau} K(x)$ almost decreases for $x \geqslant b$, there exists by definition a real number $M \geqslant 1$ such that

$$\frac{y^{-\tau} K(y)}{x^{-\tau} K(x)} \leqslant M \text{ for } y \geqslant x \geqslant b.$$

By putting $y=xt$, this inequality becomes

$$\frac{K(tx)}{K(x)} \leqslant M t^{\tau} \quad \text{for } x \geqslant b \text{ and } t \geqslant 1.$$

Hence,

$$r(t) = \limsup_{x\to\infty} \frac{K(tx)}{K(x)} \leqslant M t^{\tau} \quad \text{for } t \geqslant 1,$$

and by 3° of Theorem 2

$$t^{q} \leqslant M t^{\tau} \quad \text{for } t \geqslant 1.$$

Consequently, $q \leqslant \tau$.

If $\tau > q$, then, by definition (1.12) of the upper index q, there exists a number $d > 1$ such that

$$\frac{\log r(d)}{\log d} < \tau \quad \text{i.e.} \quad r(d) < d^{\tau}.$$

Consequently, by definition of the function r, there exists $x_0 \geqslant b$ such that

$$\frac{K(dx)}{K(x)} < d^{\tau} \quad \text{for } x \geqslant x_0.$$

On the other hand, by Theorem 1, there exist real numbers $M > 0$ and $x_1 \geqslant b$ such that

$$\frac{K(sx)}{K(x)} \leqslant M \quad \text{for } x \geqslant x_1 \text{ and } 1 \leqslant s \leqslant d.$$

Let $N = \max\{1, d^{-\tau}\} = \max\limits_{1 \leqslant s \leqslant d} s^{-\tau}$ and $c = \max\{x_0, x_1\}$. If $x \geqslant c$ and $t \geqslant 1$, then $t = sd^{n}$ for some $1 \leqslant s \leqslant d$ and for some $n = 0, 1 \ldots,$ so that

$$\frac{K(tx)}{K(x)} = \frac{K(sd^{n}x)}{K(d^{n}x)} \, \frac{K(d^{n}x)}{K(d^{n-1}x)} \cdots \frac{K(dx)}{K(x)} \leqslant M (d^{\tau})^{n} =$$

$$= M (ts^{-1})^{\tau} = M s^{-\tau} t^{\tau} \leqslant M N t^{\tau}.$$

Hence, the function $x^{-\tau} K(x)$ is almost decreasing for $x \geqslant c$. Being logarithmically bounded on $[b, c]$, the same is true for $b \leqslant x \leqslant c$ and, hence, for $x \geqslant b$.

2.5. *Proof of Theorem* 4. For the proof we have to introduce the numbers

$$\text{(2.22)} \qquad \rho_{*} = \rho_{*}(K) = \liminf_{x\to\infty} \frac{\log K(x)}{\log x}, \qquad \rho^{*} = \rho^{*}(K) = \limsup_{x\to\infty} \frac{\log K(x)}{\log x}$$

for the *O-RV* function K.

Since $x^{\sigma} \preccurlyeq K(x)$ $(x \to \infty)$ for some real σ means that there exist two positive numbers M and c such that $x^{\sigma} \leqslant MK(x)$ for $x \geqslant c$, we conclude that

$$\sigma \leqslant \frac{\log M}{\log x} + \frac{\log K(x)}{\log x} \quad \text{for } x \geqslant c,$$

and, hence, $\sigma \leqslant \rho_* (K)$. Conversely, from $\sigma < \rho_* (K)$ it follows that $\log K(x)/\log x > > \sigma$ for x large enough; for these values of x one has, consequently, $x^\sigma < K(x)$ i.e. $x^\sigma \preccurlyeq K(x)$ $(x \to \infty)$. So, we proved that

$$\rho_* (K) = \sup \{\sigma \in \mathbf{R} \mid x^\sigma \preccurlyeq K(x), \quad x \to \infty\}. \tag{2.23}$$

In (2.23) one can, evidently, substitute $\preccurlyeq$ by $\lll$, i.e. $\liminf_{x\to\infty} x^{-\sigma} K(x) > 0$ by $\lim_{x\to\infty} x^{-\sigma} K(x) = \infty$.

Since $\rho^* (K) = -\rho_* (1/K)$,

$$\rho^* = \rho^* (K) = \inf \{\tau \in \mathbf{R} \mid K(x) \preccurlyeq x^\tau, \ x \to \infty\}, \tag{2.23'}$$

where $K(x) \preccurlyeq x^\tau$ can be substituted by $K(x) \lll x^\tau$, i.e. $\limsup_{x\to\infty} x^{-\tau} K(x) < \infty$ by $\lim_{x\to\infty} x^{-\tau} K(x) = 0$.

If $\sigma < \underline{\rho} (K)$, the function $x^{-\sigma} K(x)$ almost increases on I_b, and so is bounded from 0; hence, $x^\sigma \preccurlyeq K(x)$ $(x \to \infty)$. It follows, therefore, $\underline{\rho} (K) \leqslant \rho_* (K)$ and, dually, $\rho^* (K) \leqslant \overline{\rho} (K)$. So, we have the relation

$$\underline{\rho} (K) \leqslant \rho_* (K) \leqslant \rho^* (K) \leqslant \overline{\rho} (K) \tag{2.24}$$

which, by the foregoing remarks, can be written in the form

$$\lim_{x\to\infty} x^{-\sigma} K(x) = \infty \ \text{ for each } \sigma < \underline{\rho} (K)$$

and

$$\lim_{x=\infty} x^{-\tau} K(x) = 0 \ \text{ for each } \tau > \overline{\rho} (K).$$

2.6. *Proof of Theorem* 5. If $\sigma < p$, then the function $x^{-\sigma} K(x)$ almost increases on I_b, and by the definition and some properties of almost monotone functions (especially (2.2), (viii) and (vi)) this is equivalent to each of the relations (1.18), (1.19) and (1.20).

The dual statement follows in an analoguous way.

2.7. *Proof of the Representation Theorem.* To conclude, and for the sake of completeness, we give here a proof of (1.3) $\Leftrightarrow$ (1.6), via Theorem 1, which may be of some interest.

Fix a $b > 0$ such that (by Corollarys of Theorem 1) the function $\log K$ is locally bounded (and so locally integrable) on I_b. For $s > 0$ and $x \min\{1, s\} \geqslant b$ one has

$$\int_1^s \log \frac{K(bt)}{K(xt)} \frac{dt}{t} + \int_b^x \log \frac{K(st)}{K(t)} \frac{dt}{t} = \left\{ \int_b^{bs} + \int_{sx}^{x} + \int_{sb}^{sx} + \int_x^b \right\} \log K(u) \frac{du}{u} = 0.$$

On the other side

$$\int_1^s \log K(x) \frac{dt}{t} = (\log s) \log K(x),$$

so that, by addition,

$$\int_1^s \log\left\{K(x)\frac{K(bt)}{K(xt)}\right\}\frac{dt}{t}+\int_b^x \log\frac{K(st)}{K(t)}\frac{dt}{t}=(\log s)\log K(x).$$

Consequently, the function K can be represented in the form (1.6), where

$$\alpha(x)=\alpha_s(x)=\frac{1}{\log s}\left\{\int_1^s \log K(bt)\frac{dt}{t}-\int_1^s \log\frac{K(xt)}{K(x)}\frac{dt}{t}\right\}$$

and

$$\beta(x)=\beta_s(x)=\frac{1}{\log s}\log\frac{K(sx)}{K(x)}.$$

The functions α and β are bounded on I_b because (i) the function $\log\{K(tx)/K(x)\}$ is bounded for $1\leqslant t\leqslant s$ and $b\leqslant x\leqslant c$ (for each $c\geqslant b$) and (ii), by Theorem 1, there exists a number $c\geqslant b$ such that this function is bounded for $1\leqslant t\leqslant s$ and $x\geqslant c$ too. Hence (1.3) $\Rightarrow$ (1.6).

Conversely, if K is of the form (1.6) and $|\alpha(x)|\leqslant M$, $|\beta(x)|\leqslant N$ for $x\geqslant b$, one has for $s>0$, $x\geqslant b$ and $sx\geqslant b$

$$\frac{K(sx)}{K(x)}=\exp\left\{\alpha(sx)-\alpha(x)+\int_x^{sx}\beta(t)\frac{dt}{t}\right\}\leqslant\exp(2M+N|\log s|).$$

Hence (1.6) implies (1.3).

REFERENCES

[1] Karamata, J., *Sur un mode de croissance régulière des functions*, Mathematica (Cluj) 4 (1930), 38—53.

[2] Karamata, J., *Sur un mode de croissance régulière*, Bull. Soc. Math. France 61 (1933), 55—62.

[3] Seneta, E., *Regularly Varying Functions*, Lecture Notes in Mathematicas 508, Springer-Verlag, Berlin-Heidelberg-New York 1976.

[4] Avakumović, V. G., *On a O-inverse theorem* (in Serbian), Rad Jugoslovenske Akademije Znanosti i Umjetnosti, t. 254 (Razreda Matematičko-Prirodoslovnoga 79) (1936), 167—186.

[5] Karamata, J., *Bemerkung über die vorstehende Arbeit des Herrn Avakumović, mit näherer Betrachtung einer Klasse von Funktionen, welche bei den Inversionssätzen vorkommen*, Bull. International, de l'Académie Yougoslave, Zagreb, 29 et 30 (1935), 117—123. (This is a contracted version of: *A remark of the foregoing paper of Mr V. Avakumović* (in Serbian), Rad Jugoslovenske Akademije Znanosti i Umjetnosti, t. 254 (Razreda Matematičko-Prirodoslovnoga 79) (1936), 187—200.)

[6] Bari, N. K. and Stečkin, S. B., Best approximation and differential properties of two conjugate functions (in Russian), Trudy Mosk. Mat. Obšč. 5 (1956), 483—522.

Proleterskih brigada 62
11000 Beograd (Yugoslavia)

Omladinskih brigada 212/21
11070 Beograd (Yugoslavia)