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INTERIOR VALUE PROBLEM OF HEAT CONDUCTION FOR A FINITE CIRCULAR CYLINDER

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Abstract

An interior value problem of transient heat conduction in a finite circular cylinder, being a function of both time and positition, has been solved using the technique of integral transforms i.e. with the help of finite Hankel transform and Laplace transform.

1. Introduction

Most of the problems in the theory of heat conduction require the determination of conditions at interior points when boundary conditions, such as temperature or heat flux, are prescribed at the outside surface. Such problems have been termed 'direct problems'. Correspondingly in the interior value problem or inverse problem of transient heat conduction it is required to determine the temperature or heat flux at the surface when the temperature or heat flux at an interior location is prescribed. Such problems arise in quenching studies [12, 13], in the measurement of aero-dynamic heating, and in direct calorimetry device for laboratory use [11].

For direct problem the well-known methods [2] may be applied in as straight forward manner. For interior value problem or inverse problem, however, special methods are employed. G. Stolz. Jr. [14] obtained an integral equation, and outlined a numerical method for solving inverse problems, with special reference to sphere. The problem occurred as a part of quenching programme. T. J. Mirsepassi [7] solved the problem by a graphical method. A. V. Masket and A. C. Vastano [6] solved similar problems of Mathematical Physics, using Laplace Transform and separation of variables, and termed as 'Interior Value Problems'. O. R. Burggraf [1] has obtained solution as a rapidly convergent series, with lumped capacitance approximation, as leading term. Burggraf has taken boundary conditions etc. as a function of time only. E. M. Sparrow, A. Haji Sheikh, and T. S. Lundgren [11] have also tackled the inverse problems. Sabherwal [9] has investigated the temperature distribution on the curved surface of a finite circular cylinder, when the temperature distribution on the interior surface of the cylinder is given and employed the technique of integral transforms.

The above authors have only investigated the temperature distribution on the curved surfaces of the finite or infinite cylinder by different techniques, but it appears no attempt has been made so far to determine the temperature distribution on one of the plane ends of the finite circular cylinder, when the temperature distribution is given on any interior plane normal to the axis of the cylinder and the other plane end of the cylinder. The solution of this type of interior value problem is useful in finding the temperature that should be maintained at one of the ends; the temperature at the other being known, so as to have a required temperature at one of the interior planes of the cylinder at a particular instant.

The study of literature in connection with the heat engines of various kinds clearly indicates that the cylindrical solids have an important role to play and therefore a study of temperature variation of these cylindrical solids which are used in the working of compound engines [5; p. 220], air compressor [5; p. 104], steam engine [5; p. 223], and internal combustion engine [5; p. 379] will be of great use.

During a critical study of [4] regarding the temperature variations in the solid cylindrical fuel elements [4; p. 200] and cylindrical reactor core [4; p. 214], it is observed that the study has been limited to steady state conditions when the temperature distribution is a function of radius only. We feel that it will be useful to investigate the temperature distribution in the transient case, when then effect of the difference of temperature at the two ends also effects the situation.

In this paper an interior value problem of transient heat conduction in a finite circular, being a function of both time and position, has been solved using finite Hankel Transform and Laplace Tronsform.

2. Statement of the problem

Consider the radial and axial heat flow in a finite circular cylinder bounded by surfaces $z=0$, $z=h$, and $r=a$, and initially at a temperature zero.

Mathematically the problem is formulated as:

$$\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right),\quad 0<r<a,\ 0<z<h,\ t>0. \tag{2.1}$$

Where k is the diffusivity, subject to the conditions.

$$\begin{aligned} &u(r,\ 0,\ t)=s(r,\ t),\ \text{unknown};\\ &u(r,\ z,\ 0)=u(r,\ h,\ t)=u(a,\ z,\ t)=0;\\ &u(r,\ \xi,\ t),\ \text{known},\ 0<\xi<h \end{aligned} \tag{2.2}$$

3. Solution of the problem

Applying the Hankel transform [10: p. 83] with respect to r, defined as

$$u(a_n,\ z,\ t)=\int_0^a r\,u(r,\ z,\ t)\,J_0(ra_n)\,dr \tag{3.1}$$

Where a_n is a root of the equation.

$$J_0(a\alpha_n)=0$$

to equation (2.1) and (2.2) we get

$$\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{\partial z^2}-a_n^2 u\right) \tag{3.2}$$

with $u(a_n, o, t)=s(a_n, t)$, unknown;

$$\begin{aligned} &u(a_n, z, 0)=u(a_n, h, t)=0; \\ &u(a_n, \xi, t), \text{ known.} \end{aligned} \tag{3.3}$$

Further, applying the Laplace transform with respect to t given by

$$\bar{u}(a_n, z, p)=\int_0^\infty u(a_n, z, t)\exp(-pt)\,dt, \tag{3.4}$$

to equations (3.2) and (3.3), we obtain

$$\frac{d^2\bar{u}}{dz^2}-(a_n^2+p/_k)\bar{u}=0 \tag{3.5}$$

with

$$\begin{aligned} &\bar{u}(a_n, 0, p)=\bar{s}(a_n, p), \text{ unknown;} \\ &\bar{u}(a_n, h, p=0; \\ &\bar{u}(a_n, \xi, p) \text{ known.} \end{aligned} \tag{3.6}$$

The solution of (3.5) with condition (3.6) will be

$$\bar{u}(a_n, z, p)=\bar{s}(a_n, p)\frac{\sin h\left[(h-z)\left(a_n^2+\frac{p}{k}\right)^{\frac{1}{2}}\right]}{\sin h\left[h\left(a_n^2+\frac{p}{k}\right)^{\frac{1}{2}}\right]}$$

The inverse Laplace transform of

$$\frac{\sin h\left[(h-z)(a_n^2+p/_k)^{\frac{1}{2}}\right]}{\sin h\left[h(a_n^2+p/_k)^{\frac{1}{2}}\right]}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\sin h\left[(h-z)(a_n^2+p/_k)^{1/2}\right]}{\sin h\left[h(a_n^2+p/_k)^{1/2}\right]}e^{pt}\,dp \tag{3.7}$$

The integrand has simple poles at $p=-k\dfrac{(a_n^2h^2+m^2\pi^2)}{h^2}$;

$m=0, 1, 2, \ldots \infty$, which give rise to residues

$$\frac{2\pi km}{h^2}\sin\left(\frac{m\pi z}{h}\right)\exp\left[-\frac{(a_n^2h^2+m^2\pi^2)}{h^2}kt\right]$$

Combining these with the inverse of $\bar{s}(a_n, p)$, namely $s(a_n, t)$ by the convolution theorem [3; p. 37] summing, we have

$$(3.8)\qquad u(a_n, z, t)=\frac{2\pi k}{h^2}\sum_{m=1}^{\infty} m\sin\left(\frac{m\pi z}{h}\right)\times\int_0^t s(a_n, T)\exp\left[-\frac{(a_n^2 h^2+m^2\pi^2)k(t-T)}{h^2}\right]dT$$

From (3.7) with $z=\xi$ we get

$$(3.9)\qquad \bar{s}(a_n, p)=\bar{u}(a_n, \xi, p)\frac{\sin h\,[h(a_n^2+{}^p/_k)^{1/2}]}{\sin h[(h-\xi)(a_n^2+{}^p/_k)^{1/2}]}$$

Now taking inverse Laplace transform of (3.9), we have

$$(3.10)\qquad s(a_n, t)=\frac{2\pi k}{(h-\xi)^2}\sum_{m=1}^{\infty}(-1)^{m+1} m\sin\left(\frac{m\pi h}{h-\xi}\right)\times\int_0^t u(a_n, \xi, T)\exp\left[-\frac{\{a_n^2(h-\xi)^2+m^2\pi^2\}k(t-T)}{(h-\xi)^2}\right]dT$$

Further applying inverse finite Hankel transform [10; p. 83] to equation (3.9), we obtain the solution of the interior value problem

$$(3.11)\qquad s(r, t)=\frac{4\pi k}{a^2(h-\xi)^2}\sum_n\frac{J_0(r a_n)}{J_1(a a_n)^2}\sum_{m=1}^{\infty}(-1)^{m+1} m\sin\left(\frac{m\pi h}{h-\xi}\right)\times\int_0^t u(a_n, \xi, T)\exp\left[-\frac{\{a_n^2(h-\xi)^2+m^2\pi^2\}k(t-T)}{(h-\xi)^2}\right]dT$$

Where the summation $\sum_n$ is over the positive roots of the equation

$$J_0(a a_n)=0$$

The corresponding solution of the boundary value problem will be obtained by applying the inverse Hankel Transfer [10; p. 83] to (3.8)

$$(3.12)\qquad u(r, z, t)=\frac{4\pi k}{a^2 h^2}\sum_n\frac{J_0(r a_n)}{J_1(a a_n)^2}\sum_{m=1}^{\infty} m\sin\left(\frac{m\pi z}{n}\right)\times\int_0^t s(a_n, T)\exp\left[-\frac{(a_n^2 h^2+m^2\pi^2)k(t-T)}{h^2}\right]dT$$

Where the summation $\sum_n$ is over the positive roots of the equation

$$J_0(a a_n)=0$$

REFERENCES

[1] Burggraf. O. R., *J. Heat transfer*. Trans. Amer. Mech. Engrs; 86 (1964) pp. 373.

[2] Carslaw H. S. & Jaeger. J. C. *Conduction of heat in solids*, Oxford Univ. Press London (1959).

[3] Churchill R. V., *Operational Mathematics*, McGraw Hill, New York (1958).

[4] El-Wakil. M. M., *Nuclear Power Engineering*, McGraw Hill New York (1962).

[5] Lewitt. E. H. *Theromodynamics applied to Heat Engines*, London (1953).

[6] Masket. A. V. & Vastano. A. C., Amer. J. Phys; 30 (1962) pp. 796.

[7] Mirsepassi T. J., J. Brit. Chem. Engg. I (March, 1959).

[8[Reddick H. W. & Miller F. H., *Advanced Mathematics for Engineers* (1959).

[9] Sabherwal K. C., Indian J. Pure, Appl. Phys. 3 (10) (1965) pp. 397.

[10] Sneddon I. N., *Fourier Transforms*, McGraw Hill, New York (1961).

[11] Sparrow, E. M. Haji-Sheikh, A & Lungren, T. S., J. Appl. Mech. Trans. Amer. Soc. Mech. Engrs; 31-E (1964) pp. 369.

[12] Stolz (Jr), G & Pashkis, V., J. Metals 8 (1956) pp. 1074.

[13] Stolz (Jr), G & Paschkis, V., *Iron Age*, 178 (1956) pp. 85.

[14] Stolz, G., Trans. Amer. Soc. Mech. Engrs. 82-C (1960) 00.20.

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