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QUASI-CONTRACTIONS IN BANACH SPACES

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0. Let (M, d) be a metric space and T a selfmapping of M into itself. If T satisfies the condition

$$\text{(A)} \qquad d(Tx, Ty) \leqslant q \cdot d(x, y)$$

with $q<1$ $(q=1)$, then T is called a *contraction* (*non-expansive*) *mapping*. Banach contraction mapping principle states that if M is complete and T a contraction mapping, then T has a unique fixed point.

Browder [2], Göhde [7] and Kirk [9] independently have proved that if M is a closed bounded and convex subset of a uniformly convex Banach space, then every non-expansive selfmapping has at least one fixed point.

Goebel and Zlotkiewicz [5] have proved that if M is closed and convex subset of a Banach space and T satisfies (A) with $0 \leqslant q < 2$ and T^2 is identity mapping, then T has at least one fixed point.

Many authors have discovered new classes of maps which have fixed points as it is the case with contractive or non-expansive mappings. In [3] a *quasi--contraction* was introduced as a map T of a metric space M into itself which satisfies the following condition:

$$\text{(B)} \qquad d(Tx, Ty) \leqslant q \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\};$$
$$0 \leqslant q < 1.$$

A quasi contraction has a unique fixed point, say u and $\lim_{n\to\infty} T^n x = u$ for any x in M.

Goebel, Kirk and Shimi [6] have extended a result of [2], [7] and [9] to maps which satisfy the condition

$$d(Tx, Ty) \leqslant a \cdot d(x, y) + b\,[d(x, Tx) + d(y, Ty)] + c\,[d(x, Ty) + d(y, Tx)]$$

where $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$ and $a+2b+2c<1$. I. Massabo [10] has obtained a similar result for a wider class of generalized non-expansive mappings, i.e., mappings such that

$$d(Tx, Ty) \leqslant \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\},$$

assuming that T is densifying,

For quasi-non-expansive mappings a similar result is not valid, even i M is compact (see our example 1.)

In the present paper we shall extend the class of quasicontractions weaking the condition (B) by conditions of the type

$$d(Tx, Ty) \leqslant q \cdot \max\{2\,d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\};\quad 0 \leqslant q < 1.$$

Fixed point theorems which are offered here generalize fixed point theorems of Goebel and Zlotkiewicz [5] and a result of Iseki [8].

1. Before we state our theorems we shall give an example which shows that the condition

$$\text{(B}'\text{)}\quad d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\},\quad x \neq y$$

does not ensure the existence of a fixed point of T, even when M is a compact uniformly convex Banach space.

Example 1. Let $M = [-1, 1]$ be a subset of reals and let T be a selfmapping on M defined by $Tx = \frac{x}{2}$ for $x \neq 0$ and $T(0) = 1$. Then $d(To, Tx) = 1 - \frac{x}{2} < 1 = d(o, To)$ if $x > 0$, $d(To, Tx) < 1 - x = d(x, To)$ if $x < 0$. Therefore, T satisfies (B'), as $d(Ty, Tx) = \frac{1}{2}\,d(x, y)$ for $x \cdot y \neq 0$. But T is without a fixed point.

Now we shall prove our main result:

Theorem 1. *Let X be a closed and convex subset of a Banach space and let T be a selfmapping of X into itself which satisfies the condition*

$$\text{(C)}\qquad d(Tx, Ty) \leqslant q \cdot \max\{2\,d(x\ y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all x, y in X and $0 \leqslant q < 1$. If in addition

$$(1)\qquad q^a\, d(x, y) \leqslant d(T^2 x, y) \leqslant q^b\, d(x, y)$$

for any $x \in X$ and $y \in \{Tx, Fx, TFx\}$, where $Fx = \frac{1}{2}(x + Tx)$ and $0 \leqslant a,\ b \leqslant 0$ and $a - b < 1$, then T has at least one fixed point and for any x in X a sequence $\{F^n x\}_{n=0}^{n=\infty}$ converges to some fixed point of T.

Proof. By definition of F we have

$$(2)\qquad d(x, Tx) = \|x - Tx\| = 2\left\| x - \frac{Tx + x}{2} \right\| = 2\,d(x, Fx),$$

$$(3)\qquad d(Tx, Fx) = \left\| Tx - \frac{Tx + x}{2} \right\| = \left\| \frac{Tx + x}{2} - x \right\| = d(x, Fx).$$

Put

$$u=2\,(Fx-TFx)+TFx=2\,\frac{Tx+x}{2}-TFx=(x-TFx)+Tx.$$

Then, using (2), we have

$$d\,(u,\,TFx)=\|\,2\,(Fx-TFx)\,\|=2\,d\,(Fx,\,TFx)=4\,d\,(Fx,\,F^2\,x).$$

Since

$$d(u,\,TFx)\leqslant d(u,\,Tx)+d\,(Tx,\,TFx)=\|\,(x-TFx)+Tx-Tx\,\|+d\,(Tx,\,TFx)=$$
$$=d\,(x,\,TFx)+d\,(Tx,\,TFx)\leqslant 2\cdot\max\,\{d\,(x,\,TFx),\,d\,(Tx,\,TFx)\},$$

we obtain

(4) $$d\,(Fx,\,F^2\,x)=\frac{1}{4}\,d\,(u,\,TFx)\leqslant\frac{1}{2}\max\,\{d\,(x,\,TFx),\,d\,(Tx,\,TFx)\}.$$

First we assume that

(5) $$d(x,\,TFx)\leqslant d\,(Tx,\,TFx).$$

Then by (C)

$$d(Tx,\,TFx)\leqslant q\cdot\max\,\{2\,d\,(x,\,Fx),\,d\,(x,\,Tx),\,d\,(Fx,\,TFx),\,d(x,\,TFx),\,d\,(Fx,\,Tx)\}$$

and using (2), (3) and (5) and noting that $q<1$ one has

$$d\,(Tx,\,TFx)\leqslant q\cdot\max\,\{2\,d\,(x,\,Fx),\,2\,d\,(Fx,\,TFx)\}.$$

Therefore, (5) and (4) imply

(6) $$d(Fx,\,F^2\,x)\leqslant\frac{1}{2}\,d(Tx,\,TFx)\leqslant q\cdot\max\,\{d\,(x,\,Fx),\,d(Fx,\,F^2\,x)\}.$$

Assume now that

(7) $$d\,(Tx,\,TFx)<d\,(x,\,TFx).$$

Then by (1) and (C)

$$d\,(x,\,TFx)\leqslant q^{-a}\,d\,(TTx,\,TFx)$$
$$\leqslant q^{-a}\cdot q\max\{2d(Tx,Fx),d(Tx,TTx),d(Fx,TFx),d(Tx,TFx),d(Fx,TTx)\}$$
$$\leqslant q^{1-a}\max\,\{2\,d\,(Tx,\,Fx),\,q^b\,d\,(Tx,\,x),\,d\,(Fx,\,TFx),\,d\,(Tx,\,TFx),\,q^b\,d\,(Fx,\,x)\}$$

and using (3), (2) and (7) and noting that $q^{1-a}<1$ and $q^b\geqslant 1$ we obtain

$$d\,(x,\,TFx)\leqslant q^{1-a}\max\,\{2\,d\,(x,\,Fx),\,2\,q^b\,d\,(x,\,Fx),\,2\,d\,(Fx,\,F^2\,x),\,q^b\,d\,(x,\,Fx)\}$$
$$\leqslant q^{1-a}\max\,\{2\,q^b\,d\,(x,\,Fx),\,2\,q^b\,d\,(Fx,\,F^2\,x)\}.$$

Hence and by (4) and (7) one has

(8) $$d\,(Fx,\,F^2x)\leqslant\frac{1}{2}\,d\,(x,\,TFx)\leqslant q^{1-a+b}\max\,\{d\,(x,\,Fx),\,d(Fx,\,F^2x)\}.$$

Therefore, from (6) and (8) follows that (4) implies

(10) $$d\,(Fx,\,F^2x)\leqslant q^{1-a+b}\max\,\{d\,(x,\,Fx),\,d(Fx,\,F^2x)\},$$

since $q \leqslant q^{1-a+b}$. If $d(Fx, F^2x) > 0$, then (10) reduces in

$$d(Fx, F^2x) \leqslant q^{1-a+b}\, d(x, Fx), \tag{11}$$

as $q^{1-a+b} < 1$. It is clear that (11) is valid in the case $d(Fx, F^2x) = 0$.

Since $q^{1-a+b} < 1$, by (11) the sequence $\{F^n x\}_{n=}^{n=\infty}$ is the Cauchy sequence. By the completeness of X, there exists some element z in X such that $\lim F^n x = z$.
As

$$d(z, Tz) \leqslant d(z, F^{n+1}x) + \left\| \frac{TF^n x + F^n x}{2} - Tz \right\|$$

$$\leqslant d(z, F^{n+1}x) + \frac{1}{2} d(F^n x, Tz) + \frac{1}{2} d(TF^n x, Tz) \leqslant d(z, F^{n+1}x) + \frac{1}{2} d(F^n x, Tz) +$$

$$\frac{1}{2} q \max\{2\, d(F^n x, z), d(F^n x, TF^n x), d(z, Tz), d(z, TF^n x), d(F^n x, Tz)\}$$

$$\leqslant d(z, F^{n+1}x) + \frac{1}{2} d(F^n x, Tz) + \frac{q}{2} \max\{2\, d(F^n x, z), 2\, d(F^n x, F^{n+1}x), d(z, Tz),$$

$$[d(z, F^n x) + d(F^n x, TF^n x], d(F^n x, Tz)\},$$

we obtain, letting n tends to infinity,

$$d(z, Tz) \leqslant \frac{1}{2} d(z, Tz) + \frac{q}{2} d(z, Tz) = \frac{1+q}{2} d(z, Tz).$$

Hence $d(z, Tz) = 0$, i.e. $Tz = z$ and the proof of the Theorem is complete

Corollary 1 (Goebel and Zlotkiewicz [5], th. 1). *If C is a closed and convex subset of a Banach space and if $T: C \to C$ satisfies conditions:* 1^0 $T = I$, 2^0 $\|Tx - Ty\| \leqslant K \|x - y\|$, *where* $0 \leqslant K < 2$, *then T has at least one fixed point.*

Proof follows immediately from the Theorem 1, as 1^0 implies that our condition (1) is satisfied with $a = b = o$ and 2° implies (C).

We shall give an example where T satisfies 2^0 of the corollary and (1) of the theorem 1, but $T^2 \neq I$.

Example 2. Let $X = R$ be a set of reals with $d(x, y) = |x - y|$ and let a mapping $T: R \to R$ be defined by

$$Tx = -x, \text{ if } x \geqslant 0$$
$$= -1{,}2 \cdot x, \text{ if } x < 0.$$

Then

$$d(Tx, Ty) = d(x, y), \text{ if } x \geqslant 0,\ y \geqslant 0,$$

$$d(Tx, Ty) = 1{,}2\ d(x, y), \text{ if } x < 0,\ y < 0,$$

$$d(Tx, Ty) < 1{,}2\,|x| + 1{,}2\,|y| = 1{,}2\, d(x, y), \text{ if } x \cdot y < 0$$

Therefore, 2^0 is satisfied with $k=1{,}2$, or (C) in the theorem 1 with $q=\frac{k}{2}=0{,}6$. Since $d(x,y)\leqslant d(T^2x,y)$ $(y\in\{Tx, Fx, TFx\})$, and

$$\max\{d(T^2x,y)\cdot[d(x,y)]^{-1}: y\in\{Tx, Fx, TFx\}\}$$

is attained for $y=TFx$ $(x\neq 0)$ the relation (1) will follow if we show that $d(T^2x, TFx)\leqslant(0{,}6)^b\, d(x, TFx)$ for $-1<b\leqslant 0$. But, this relation is fulfilled for $b=-\frac{1}{2}$, as

$$d(T^2x, TFx)=1{,}1\,x \quad (=1{,}2\,x) \quad \text{for} \quad x<0 \quad (x>0) \quad \text{and} \quad d(x, TFx)=0{,}9x \quad (=x)$$

for $x<0$ $(x>0)$ and hence $d(T^2x, TFx)\,[d(x, TFx)]^{-1}\leqslant\frac{1{,}1}{0{,}9}<(0{,}6)^{-1/2}$.

Corollary 2. (Iseki [8]). *If $T: X\to X$ is such that $T^2=I$ and*

(D) $\quad d(Tx, Ty)\leqslant\alpha\, d(x,y)+\beta\,[d(x, Tx)+d(y, Ty)]$; $\alpha\geqslant 0$, $\beta\geqslant 0$; $\alpha+4\beta<2$,

then T has at least one fixed point.

Proof. Since

$$\frac{\alpha}{2}\,2\,d(x,y)+2\beta\,\frac{1}{2}\,[d(x, Tx)+d(y, Tx)]\leqslant\left(\frac{\alpha}{2}+2\beta\right)\max\left\{2\,d(x,y),\right.$$
$$\left.\frac{1}{2}\,[d(x, Tx)+d(y, Ty)]\right\}$$
$$\leqslant\left(\frac{\alpha}{2}+2\beta\right)\max\{2\,d(x,y), d(x, Tx), d(y, Ty)\}$$
$$\leqslant\left(\frac{\alpha}{2}+2\beta\right)\max\{2\,d(x,y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\},$$

the proof follows from our Theorem if we put $\frac{\alpha}{2}+2\beta=q$.

Now we shall give an example of a mapping which satisfies (C), but neither (D) nor (B).

Example 3. Let $X=R$ be a set of reals with usual norm and let a mapping T of X into itself be defined by

$$Tx=-10\,x, \ \text{if} \ x\geqslant 0$$
$$=-0{,}1\,x, \ \text{if} \ x<0.$$

Then

$$x\leqslant 0 \ \text{and} \ y\leqslant 0 \ \text{imply} \ d(Tx, Ty)=0{,}1\ d(x,y),$$

$$x\geqslant 0,\ y\geqslant 0 \ \text{imply} \ d(Tx, Ty)=|10\,x-10y|\leqslant 10\,m=\frac{10}{11}\,11\,m=\frac{10}{11}\,d(m, Tm);$$

$$m=\max\{x,y\},$$

$y \leqslant 0,\ x > 0$ imply $d(Tx, Ty) = 10x + 0{,}1(-y) \leqslant 10x + \frac{1}{2}x = \frac{21}{2 \cdot 11} \cdot 11x =$

$$= \frac{21}{22} d(x, Tx), \quad \text{for} \quad -y \leqslant 5x, \quad \text{or}$$

$$d(Tx, Ty) = 10x + 0{,}1(-y) \leqslant \frac{7}{8}(2x - 2y) = \frac{7}{8} 2d(x, y), \quad \text{for} \quad -y > 5x.$$

Therefore,

$$d(Tx, Ty) \leqslant \frac{21}{11} \max\{2d(x, y), d(x, Tx), d(y, Ty)\}$$

It means that T satisfies (C) with $q = \frac{21}{22}$, for every x, y in X. To see that T does not satisfy (D), let $x \geqslant 0$ and $y > 34x$. Then

$$d(Tx, Ty) = 10y - 10x > 3(y - x) + \frac{3}{5}(11x + 11y) =$$

$$= 3d(x, y) + \frac{3}{5}[d(x, Tx) + d(y, Ty)],$$

that is,

$$d(Tx, Ty) > \alpha d(x, y) + \beta [d(x, Tx) + d(y, Ty)]$$

for $\alpha = 3$ and $\beta = \frac{3}{5}$. But $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\alpha + 4\beta < 2$ in (D) imply $\alpha < 2$ and $\beta < \frac{1}{2}$. Note that here $T^2 = I$ and X is convex Banach space. Therefore, all hypotheses of our Theorem are fulfilled.

Now we shall state a some different theorem from Theorem 1.

Theorem 2. *If X is a closed and convex subset of a Banach space and if $T: X \to X$ satisfies* (1) *and the following condition*

$$\text{(C}') \quad d(Tx, Ty) \leqslant q \max\left\{2d(x, y), d(x, Tx), d(y, Ty), \frac{2}{3}[d(x, Ty) + d(y, Tx)]\right\},$$

where $q < 1$, then T has at least one fixed point and for any x in X a sequence $\{F^n x\}_{n=0}^{\infty}$ converges to a fixed point of T.

Proof. Since proof is similar to the proof of Theorem 1, we omit details. We shall show only how the relation (6) follows in this case.

Assume that, for example, $d(x, TFx) \leqslant d(Tx, TFx)$. Then by (C′)

$$d(Tx, TFx) \leqslant q \max\left\{2d(x, Fx), d(x, Tx), d(Fx, TFx), \frac{2}{3}[d(x, TFx) + d(Fx, Tx)]\right\}$$

$$\leqslant q \max\left\{2d(x, Fx), 2d(Fx, F^2x), \frac{2}{3}[d(Tx, TFx) + d(x, Fx)]\right\}.$$

Since the case $d(Tx, TFx) \leqslant q \cdot \frac{2}{3}[d(Tx, TFx)+d(x, Fx)]$ implies

$$d(Tx, TFx) \leqslant \frac{2q}{3-2q} d(x, Fx) \leqslant 2q \cdot d(x, Fx),$$

we obtain

$$d(Tx, TFx) \leqslant 2q \cdot \max\{d(x, Fx), d(Fx, F^2 x)\}.$$

Therefore,

$$d(Fx, F^2 x) \leqslant \frac{1}{2} \cdot 2q \cdot \max\{d(x, Fx), d(Fx, F^2 x)\} = q \cdot \max\{d(x, Fx), d(Fx, F^2 x)\}$$

which is the relation (6).

In a normed space we have the following result:

Theorem 3. *Let X be a closed convex subset of a normed space an let $T: X \to X$ satisfy the condition*

$$d(Tx, Ty) \leqslant q \cdot \max\{cd(x, y), [d(x, Tx)+d(y, Ty)], [d(x, Ty)+d(y, Tx)]\},$$

where $c \geqslant 0$. If a sequence $x_{n+1}=(1-t)x_n+tTx_n$, $n=0, 1, 2, \ldots$, $x_0 \in X$, $0<t<1$ converges in X then T has a fixed point.

Proof. Let z be in X such that

$$\lim_{n \to \infty} x_{n+1} = z.$$

We shall show shat z is a fixed point of T.

$$\begin{aligned} d(z, Tz) &\leqslant d(z, x_{n+1}) + \|(1-t)x_n + tTx_n - Tz\| \\ &= d(z, x_{n+1}) + \|(1-t)x_n + tTx_n - (1-t)Tz - tTz\| \\ &\leqslant d(z, x_{n+1}) + (1-t)d(x_n, Tz) + td(Tx_n, Tz) \\ &\leqslant d(z, x_{n+1}) + (1-t)d(x_n, Tz) + \\ &+ tq \cdot \max\{cd(x_n, z), [d(x_n, Tx_n)+d(z, Tz)], [d(x_n, Tz)+d(z, Tx_n)]\} \\ &\leqslant d(z, x_{n+1}) + (1-t)d(x_n, Tz) + \\ &+ tq \cdot \max\left\{cd(x_n, z), \left[\frac{1}{t} d(x_n, x_{n+1}) + d(z, Tz)\right], \left[d(x_n, Tz) + d(z, x_n) + \right.\right. \\ &\left.\left. + \frac{1}{t} d(x_n, x_{n+1})\right]\right\}. \end{aligned}$$

Letting n to tend to infinity we obtain

$$d(z, Tz) \leqslant (1-t)d(z, Tz) + tqd(z, Tz)$$

which implies $d(z, Tz)=0$. This completes the proof of the theorem.

The following theorem extends the Theorem 3 of [5].

Theorem 4. *Let X be a closed, bounded and convex subset of a uniformly convex Banach space. If $T: X \to X$ satisfies* (C) *or* (C′) *and*

$$d(T^2 x, T^2 y) \leqslant a d(x, y) + b[d(x, T^2 x) + d(y, T^2 y)] + c[d(x, T^2 y) + d(y, T^2 x)],$$

where $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$ and $a + 2b + 2c < 1$, then T has at least one fixed point.

Proof. By the result of Goebel, Kirk and Shimi [6], a set P of fixed points of T^2 is non-void. It is easy to verify that P is closed and convex. Clearly that $T(P) = P$ and $T^2 = I$ on P. Hence we may apply Theorem 1 or Theorem 2.

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