PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 21 (35), 1977, pp. 35–40

TOPOLOGIES ON SPACES OF CONTINUOUS MULTIFUNCTIONS

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(Received January 12, 1976)

Summary

Two properties which a topology on spaces of continuous multivalued functions may have are investigated. As far as ordering of topologies is concerned, these properties are hereditary but in a sense opposite to each other. If Y is a locally compact Hausdorff space and Z an arbitrary topological space, then the unique topology on the space of all continuous multifunctions from Y to Z having both properties is the compact open topology (see [2] for analogous results about spaces of singlevalued continuous functions).

1. In what follows X, Y, Z are topological spaces. By Z^{mY} $[C^m(Y, Z)]$ we designate the set of all [continuous] multifunctions from Y to Z (for definitions about continuity of multifunctions, see [5]).

If $g: X \times Y \to Z$ is a multifunction, a single-valued function g^* may be defined as follows:

$$g^*: X \to Z^{mY}, \text{ where } g^*(x): Y \to Z$$
$$x \to g^*(x) \qquad\qquad y \to g^*(x)(y) = g(x, y).$$

Conversely, if $g^*: X \to Z^{mY}$ is a single-valued function, we may define a multifunction $g: X \times Y \to Z$ by putting $g(x, y) = g^*(x)(y)$. The functions g and g^* will be called *associated* functions, and they will always be used in this sense.

It is easily seen that the continuity of g implies the continuity of $g^*(x)$; so if g is continuous, $g^*(x) \in C^m(Y, Z)$.

2. A topology τ on $C^m(Y, Z)$ is called *admissible* if the multifunction $\omega: (C^m(Y, Z), \tau) \times Y \to Z$:

$$(f, y) \to \omega(f, y) = f(y)$$

is continuous ($(C^m(Y, Z), \tau) \times Y$ has the producttopology).

Example: The discrete topology on $C^m(Y, Z)$ is the finest admissible topology.

Proposition 1: *The following statements are equivalent:*

(i) τ *is admissible on* $C^m(Y, Z)$.

(ii) *For any topological space* X, $g^*: X \to (C^m(X, Z), \tau)$ *is continuous*

$$\Rightarrow g \text{ is continuous}$$

Proof. (i) $\Rightarrow$ (ii): Suppose g^* is continuous. We define the function

$$h: X \times Y \to C^m(Y, Z) \times Y$$
$$(x, y) \to (g^*(x), (y)).$$

Then $\omega \circ h = g$, while ω and h are continuous; hence so is their composition g.

(ii) $\Rightarrow$ (i): Since (ii) is supposed to hold for any X, we take the space $(C^m(Y, Z), \tau)$ as special X, and the identity function on $(C^m(Y, Z), \tau)$ as g^*. The associated function g is defined by $g: (C^m(Y, Z), \tau) \times Y \to Z$

$$(f, y) \to g(f, y) = g^*(f)(y) = f(y)$$

Hence $g \equiv \omega$, and by the supposed continuity of g we immediately obtain that ω is continuous. This means that τ is admissible.

Definitions. Let Δ be an upward directed set, and $\{f_\mu\}_{\mu \in \Delta}$ a net in $C^m(Y, Z)$.

—$\{f_\mu\}_{\mu \in \Delta}$ is called *upper semicontinuously convergent* (u.s.c.c) to $f \in C^m(Y, Z)$ if, for each $y \in Y$ and for each open set W in Z such that $f(y) \subset W$, there exists a $\mu' \in \Delta$ and a neighborhood V of y in Y such that $f_\mu(V) \subset W$ for all $\mu \geqslant \mu'$.

—$\{f_\mu\}_{\mu \in \Delta}$ is called *lower semicontinuously convergent* (l.s.c.c) to $f \in C^m(Y, Z)$ if, for each $y \in Y$, for each $z \in f(y)$ and for each open neighborhood W of z in Z, there exists a $\mu' \in \Delta$ and an open neighborhood V of y in Y such that $f_\mu(t) \cap W \neq \varnothing$, $\forall t \in V$, $\forall \mu \geqslant \mu'$.

—$\{f_\mu\}_{\mu \in \Delta}$ is called *continuously convergent* (c.c) to $f \in C^m(Y, Z)$ if $\{f_\mu\}_{\mu \in \Delta}$ is at the same time u.s.c.c. and l.s.c.c. to f.

Proposition 2. *Let* τ *be an admissible topology on* $C^m(Y, Z)$. *If a net* $\{f_\mu\}_{\mu \in \Delta}$ *in* $C^m(Y, Z)$ *converges to* f *in* $(C^m(Y, Z), \tau)$, *then* $\{f_\mu\}_{\mu \in \Delta}$ *is c.c. to* f.

Proof. The upward directed set Δ gives rise to a topological space $\Delta' = \Delta \cup \{\infty\}$ by adjoining an element ∞ to Δ, assuming that $\infty > \mu$ for each $\mu \in \Delta$; as open sets we take all singletons of Δ, and all sets of the form $N_{\mu'}^{\infty} = \{\mu \in \Delta' : \mu \geqslant \mu'$, where $\mu' \in \Delta\}$. (Δ' will play the role of the topological space X in our earlier notation). On Δ' we define the single-valued function g^* with values in $(C^m(Y, Z), \tau)$ by

$$\begin{cases} g^*(\mu) = f_\mu & (\mu \in \Delta), \\ g^*(\infty) = f. \end{cases}$$

It is easily verified that g^* is continuous. Hence, by assumption and prop. 1, the associated multifunction g is continuous, where

$$g: \Delta' \times Y \to Z$$
$$\begin{cases} (\mu, y) \to g(\mu, y) = g^*(\mu)(y) = f_\mu(y) & (\mu \in \Delta), \\ (\infty, y) \to g(\infty, y) = g^*(\infty)(y) = f(y) \end{cases}$$

Expressing first the l.s.c. of g at the fixed point ∞ of Δ' and an arbitrary point y of Y we obtain: if $z \in g(\infty, y)$, and W is an open set in Z such that $z \in W$, there exists a neighborhood $N^{\infty}_{\mu'}$, of ∞ in Δ' and an open neighborhood V of y in Y such that $g(\mu, t) \cap W \neq \varnothing$, for all $\mu \in N^{\infty}_{\mu'}$, and all $t \in V$; or $f_\mu(t) \cap W \neq \varnothing$, $\forall \mu \geqslant \mu'$, $\forall t \in V$.

Hence $\{f_\mu\}_{\mu \in \Delta}$ is l.s.c.c. to f in $C^m(Y, Z)$. The proof that $\{f_\mu\}_{\mu \in \Delta}$ is u.s.c.c. to f is analogous.

3. A topology τ on $C^m(Y, Z)$ is called *proper* if the following condition is true:

for any topological space X, the continuity of the multifunction $g: X \times X \to Z$ implies the continuity of the associated function $g^*: X \to (C^m(Y, Z), \tau)$.

Example: The trivial topology on $C^m(Y, Z)$ is the coarsest proper topology.

Proposition 3. *The following conditions are equivalent:*

(i) τ *is proper on* $C^m(Y, Z)$

(ii) *each net* $\{f_\mu\}_{\mu \in \Delta}$ *in* $C^m(Y, Z)$ *that c.c. to* f *in* $C^m(Y, Z)$ *is convergent to* f *in* $(C^m(Y, Z), \tau)$.

Proof. (i) $\Rightarrow$ (ii): Let $\{f_\mu\}_{\mu \in \Delta}$ be a net that c.c. to f in $C^m(Y, Z)$. As in the proof of prop. 2, we use the topological space Δ' and we define the multifunction g as follows:

$$g: \Delta' \times Y \to Z$$

$$\begin{cases} (\mu, y) \to g(\mu, y) = f_\mu(y) \\ (\infty, y) \to g(\infty, y) = f(y) \end{cases}$$

A straightforward verification (using the continuity of the functions f_μ and the continuous convergence of the net $\{f_\mu\}$) shows that g is continuous. By assumption the associated single valued function $g^*: \Delta' \to (C^m(Y, Z), \tau)$ is continuous, in particular at the element ∞ of Δ'. Since $\{\mu\}_{\mu \in \Delta}$ is a net in Δ' that converges to the point ∞, the net $\{g^*(\mu)\}_{\mu \in \Delta}$ will converge to $g^*(\infty)$ in $(C^m(Y, Z), \tau)$. This means that $\{f_\mu\}_{\mu \in \Delta}$ is convergent to f in $(C^m(Y, Z), \tau)$.

(ii) $\Rightarrow$ (i): Let X be an arbitrary topological space. We have to show that, if g is a continuous multifunction from $X \times Y$ to Z, the associated single-valued function $g^*: X \to (C^m(Y, Z), \tau)$ is continuous. This will be the case if the following condition is satisfied: if $x \in X$, and $\{x_\mu\}_{\mu \in \Delta}$ is an arbitrary net in X that converges to x, then $\{g^*(x_\mu)\}$ converges to $g^*(x)$. By our assumptions it suffices to show that $\{g^*(x_\mu)\}_{\mu \in \Delta}$ is continuously convergent to $g^*(x)$. As usual we just show one part of this statement, e.g. the u.s.c.c. (the other part being analogous). So let $y \in Y$, W an open set in Z such that $g^*(x)(y) = (= g(x, y)) \subset W$. Since g is u.s.c., there exist neighborhoods U of x in X, V of y in Y, such that $g(U \times V) \subset W$. By convergence of the net $\{x_\mu\}_{\mu \in \Delta}$ to x, there exists a $\mu' \in \Delta$ such that $x_\mu \in U$ for all $\mu \geqslant \mu'$. Hence $g(x_\mu, t) \subset W$, $\forall \mu \geqslant \mu'$, $\forall t \in V$; or $g^*(x_\mu)(V) \subset W$, $\forall \mu \geqslant \mu'$, which shows the u.s.c.c.

Proposition 4. *Let* σ *and* τ *be two topologies on* $C^m(Y, Z)$. *Then:*

(i) *If* σ *is admissible and* $\tau \geqslant \sigma$, *then* τ *is admissible*

(ii) *If* τ *is proper and* $\tau \geqslant \sigma$, *then* σ *is proper*

(iii) *If* τ *is proper and* σ *is admissible, then* $\tau \leqslant \sigma$.

Proof. (i) and (ii) are immediate. For (iii), take $(C^m(Y, Z), \sigma)$ as the space X to express that the topology τ is proper.

Corollary 1. There can be no more than one topology on $C^m(Y, Z)$ which is at the same time admissible and proper.

Corollary 2. If there exists an admissible and proper topology on $C^m(Y, Z)$, it is the coarsest admissible one and the finest proper one.

4. In [2], admissible and proper topologies on spaces of continuous single--valued functions have been defined; roughly speaking one could say that our definitions and theorems are the multivalued versions of those appearing in [2].

Using the notations and definitions of [4], a multifunction h from Y to Z may also be viewed as a single-valued function from Y to $\mathcal{A}(Z)$ ($\mathcal{A}(Z) = \{E \subset Z : E \neq \varnothing\}$); the multifunction h is continuous if and only if h is a continuous single-valued function from Y to $\mathcal{A}(Z)$, when $\mathcal{A}(Z)$ is provided with the finite topology. We write shortly Z_F to denote $\mathcal{A}(Z)$ with the finite topology, and h_F for h considered as the single-valued function corresponding to h. It is then clear that $C^m(Y, Z)$ is bijective with $C(Y, Z_F)$, that a topology τ on $C^m(Y, Z)$ induces a topology τ_F on $C(Y, Z_F)$ (0 open in $(C(Y, Z_F), \tau_F)$ iff 0 is open in $(C^m(Y, Z), \tau)$), and so on.

Proposition 5. *A topology* τ *on* $C^m(Y, Z)$ *is admissible* [*proper*] *if and only if the topology* τ_F *on* $C(Y, Z_F)$ *is admissible* [*proper*].

Proof. We just prove that τ admissible on $C^m(Y, Z)$ implies τ_F admissible on $C(Y, Z_F)$, the other implications being analogous.

Let X be an arbitrary topological space, g_F^* a continuous function from X to $C(Y, Z_F)$. We have to show that the associated (single-valued) function g_F from $X \times Y$ to Z_F is continuous.

Let $<U_1, \ldots, U_n>$ [4] be a basis open set containing the "point" $g_F(x_0, y_0)$ in Z_F $(x_0 \in X, y_0 \in Y)$. The function g^* from X to $C^m(Y, Z)$ corresponding to g_F^* is continuous, and by admissibility of τ the same is true for the corresponding multifunction g from $X \times Y$ to Z; furthermore $g(x_0, y_0) \subset \bigcup_{i=1}^{n} U_i$ and $g(x_0, y_0) \cap U_i \neq \varnothing$, $\forall i \in \{1, 2, \ldots, n\}$. Expressing the upper and lower semi--continuity respectively of g we find:

— there exist neighborhoods V_1 of x_0, W_1 of y_0 such that

$$g(x, y) \subset \bigcup_{i=1}^{n} U_i, \ \forall x \in V_1, \ \forall y \in W_1.$$

— for each $i \in \{1, 2, \ldots, n\}$, there exist neighborhoods V_2^i of x_0, W_2^i of y_0 such that $g(x, y) \cap U_i \neq \varnothing$, $\forall x \in V_2^i$, $\forall y \in W_2^i$.

Putting $V=V_1\cap\left(\bigcap_{i=1}^{n} V_2{}^i\right)$, $W=W_1\cap\left(\bigcap_{i=1}^{n} W_2{}^i\right)$, we obtain neighborhoods V and W of x_0 and y_0 respectively such that $g_F(x, y)\in<U_1, \dots, U_n>$, $\forall x\in V$, $\forall y\in W$, showing the continuity of g_F.

From prop. 5 it follows that we equally well could have defined admissible and proper topologies on $C^m(Y, Z)$ by constructing the corresponding space $C(Y, Z_F)$, and then using the definitions for single-valued functions. In particular this procedure may be used to show that the following converse of prop. 2 is true:

Proposition 2'. *If each net $\{f_\mu\}_{\mu\in\Delta}$ in $C^m(Y, Z)$ that converges to f in $(C^m(Y, Z), \tau)$ is continuously convergent to f, then τ is admissible.*

5. Let Y and Z be topological spaces; $A\subset Y$, $U\subset Z$, $V\subset Z$. We define $(A; U, V)=\{f\in C^m(Y, Z): f(A)\subset U$ and $f(a)\cap V\neq\varnothing$ for each $a\in A\}$. Given a family $\mathcal{A}$ of subsets of Y, we define the "set-open" (or $\mathcal{A}$-open) topology on $C^m(Y, Z)$ as the one having for subbase the sets $(A; U, V)$, $A\in\mathcal{A}$, U and V open in Z.

Proposition 6. *If all sets of the family $\mathcal{A}$ are compact, then the $\mathcal{A}$-open topology on $C^m(Y, Z)$ is proper.*

Proof. X being an arbitrary topological space, and g a continuous multifunction from $X\times Y$ to Z, we have to show that the associated function g^* is continuous at an arbitrary point x_0 of X. To this end it is sufficient to show: if $(A; U, V)$ is a subbase member containing $g^*(x_0)$, there exists a neighborhood W of x_0 in X such that $g^*(W)\subset(A; U, V)$. Using the relation between g and g^* and the definition of $(A; U, V)$, this may also be stated as follows: if $g(x_0, A)\subset U$ and $g(x_0, a)\cap V\neq\varnothing$ for each $a\in A$, there exists a neighborhood W of x_0 such that $g(w, A)\subset U$ and $g(w, a)\cap V\neq\varnothing$ for each $a\in A$ and each $w\in W$.

Using the u.s.c. of g we obtain: for each point $(x_0, a)\in\{x_0\}\times A$ there exist open sets M_a containing x_0 and N_a containing a such that $g(M_a\times N_a)\subset U$. The compactness of $\{x_0\}\times A$ gives rise to a finite open covering $M_{a_i}\times N_{a_i}$ ($i=1, \dots, n$) of $\{x_0\}\times A$. Putting $M=\bigcap_{i=1}^{n} M_{a_i}$, we obtain a neighborhood M of x_0 such that $g(m, A)\subset U$ for each $m\in M$.

In the same fashion, using the l.s.c. of g, the compactness of $\{x_0\}\times A$ and V as open set in Z, we can show the existence of a neighborhood N of x_0 such that $g(n, a)\cap V\neq\varnothing$ for each $n\in N$ and each $a\in A$. Putting $W=M\cap N$ we get the required result.

Example: the compact open topology.

The compact open topology (or c-topology) is that set-open topology for which $\mathcal{A}$ is the family of all compact sets (see [6]). Hence the c-topology is the finest proper $\mathcal{A}$-open topology, starting from compact sets.

6. A family $\mathcal{A}$ of sets in Y is called a regular family if, for each $y\in Y$ and for each neighborhood U of y, there exists a set A of the family $\mathcal{A}$ such that $y\in\mathring{A}\subset A\subset U$.

Proposition 7. *An $\mathcal{A}$-open topology on $C^m(Y, Z)$, based on a regular family $\mathcal{A}$, is admissible.*

Proof. We have to prove that the multifunction ω appearing in the definition of admissibility is continuous. Once more we just show the u.s.c. of ω.

If W is an open set in Z containing $\omega(f, y)(=f(y))$, then by the u.s.c. of f there exists an open set U in Y such that $y \in U$ and $f(U) \subset W$. Since $\mathcal{A}$ is regular, there exists a set $B \in \mathcal{A}$ containing y such that $f(B) \subset W$, from which we may conclude that $f \in (B; W, W)$. The u.s.c. of ω will be proved if $\omega((B; W, W) \times \mathring{B}) \subset W$; this means, if $\omega(g, t)(=g(t)) \subset W$ for $g \in (B; W, W)$ and $t \in \mathring{B}$. But this inclusion is immediately true by definition of $(B; W, W)$.

Corollary.

If Y is a locally compact Hausdorff space, then the c-topology on $C^m(Y, Z)$ is proper and admissible.

The author would like to thank the referee for some helpful suggestions.

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