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ON MARKOV PROPERTIES OF FINITELY PRESENTED GROUPS

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1. Notions and notation

This paper deals with finitely presented (*f.p.*) groups, which will be denoted by $G, H, K, \ldots$. A formula of the predicate calculus in which G is the only free variable, will determine a property of *f.p.* groups. We restrict ourselves in the sequel to nontrivial properties of *f.p.* groups, and to algebraic ones (i.e. to those which are preserved under isomorphisms). Subgroup, proper subgroup, normal subgroup and isomorphism relation will be denoted by $<$, $|\!<$, $\lhd$ and $\cong$ respectively.

A property P is called a *Markov property* if it satisfies the following condition:

$$(\exists K)(\forall G)(\forall H)(H < G \Rightarrow \neg (P(G) \wedge H \cong K)).$$

The importance of this class of properties of *f.p.* groups derives from the following facts:

— most of the natural and well-known properties of groups, such as being trivial, finite, cyclic, Abelian, free etc., are Markov properties,

— every Markov property is algorithmically (recursively) unrecognizable [1], [8], (Note that the complement of a Markov property is also unrecogonizable.)

Let U denote a universal *f.p.* group, *i.e.* an *f.p.* group containing (as a subgroup) an isomorphic image of every *f.p.* group. The existence of such a group was proved by Higman [4]. However, it is not unique, as e.g. any free power of it, $U * U * \cdots * U$, is again a (nonisomorphic) universal *f.p.* group. The class of all universal *f.p.* groups will be denoted, in what follows, by $\bar{U}$.

2. Problems considered

To decide whether a given property P of *f.p.* groups is actually a Markov property or not, one has to find out a group K which is not isomorphic to any subgroup of an *f.p.* group G having P, or to prove that such a group K does not exist. However, the above definition of a Markov property does not suggest how and where to look for the mentioned group K, so that the above embedding problem appears to be quite difficult in general (see e.g. [2]). To illustrate this, let us consider the following properties:

P_1: being isomorphic to one of its own proper subgroups, i.e.

$$P_1(G) \Leftrightarrow (\exists H \mid< G)(H \cong G),$$

P_2: having for every *f.p.* factor-group an isomorphic subgroup,

i.e.

$$P_2(G) \Leftrightarrow (\forall N \lhd G)(\exists H < G)(H \cong G/N).$$

Although these properties are rather simple, it is not obvious whether they are Markov properties or not.

Further, it is evident that the intersection $P_1 \cap P_2$ of any two Markov properties P_1 and P_2 is again a Markov property; however, the same question cannot be immediately settled for their union $P_1 \cup P_2$. Namely, even if there exists a group K_1 which cannot be embedded into a group enjoing P_1 and an analogous group K_2 exists for P_2, it is still questionable whether such a group K exists for $P_1 \cup P_2$. Particularly, none of K_1 and K_2 must in general solve the problem, contrary to the case of $P_1 \cap P_2$.

Hence, it seems that a more efficient test to decide whether a given property is a Markov property or not would be of interest. It turns out that, to that purpose, the aforementioned universal *f.p.* groups can be made use of.

3. Theorem (*on the connection between Markov properties and universal groups*):

A nontrivial algebraic property P of f.p. groups is a Markov property if and only if none of the universal f.p. groups enjoys P.

Proof.

Necessity. By definition of universal *f.p.* group we have

$$(\forall K)(\forall U \in \bar{U})(\exists H < U)(H \cong K).$$

Consequently,

$$\begin{aligned}(\exists U \in \bar{U}) P(U) &\Rightarrow (\exists U \in \bar{U}) P(U) \wedge (\forall K)(\forall U \in \bar{U})(\exists H < U)(H \cong K)\\ &\Rightarrow (\exists U \in \bar{U})(\forall K)(\exists H)(P(U) \wedge H < U \wedge H \cong K)\\ &\Rightarrow (\forall K)(\exists U \in \bar{U})(\exists H < U)(P(U) \wedge H \cong K)\\ &\Rightarrow (\forall K)(\exists G)(\exists H < G)(P(G) \wedge H \cong K).\end{aligned}$$

Finally, by means of contraposition we obtain

$$(\exists K)(\forall G)(\forall H < G) \neg (P(G) \wedge H \cong K) \Rightarrow (\forall U \in \bar{U}) \neg P(U). \tag{1}$$

Sufficiency. Using again the definition of a universal *f.p.* group etc., one can easily verify the following statements

$$(\forall U \in \bar{U})(\forall G)(\forall H < G)(G \notin \bar{U} \Rightarrow \neg (H \cong U))$$

$$(\forall U \in \bar{U}) \neg P(U) \Rightarrow (\forall G)(\neg P(G) \vee G \notin \bar{U}),$$

so that one has

$$(\forall U\in \bar{U})\neg P(U)\Rightarrow(\forall U\in \bar{U})\neg P(U)\wedge(\forall U\in \bar{U})(\forall G)(\forall H<G)(G\notin \bar{U}\Rightarrow\neg(H\cong U))$$
$$\wedge((\forall U\in \bar{U})\neg P(U)\Rightarrow(\forall G)(\neg P(G)\vee G\notin \bar{U}))$$
$$\Rightarrow(\forall U\in \bar{U})(\forall G)(\forall H<G)(G\notin \bar{U}\Rightarrow\neg(H\cong U)\wedge(\forall G)(\neg P(G)\vee G\notin \bar{U})$$
$$\Rightarrow(\forall U\in \bar{U})(\forall G)(\forall H<G)(P(G)\Rightarrow\neg(H\cong U))$$
$$\Rightarrow(\exists K)(\forall G)(\forall H<G)\neg(P(G)\wedge H\cong U). \tag{2}$$

The proof of the above Theorem is completed by (1) and (2).

4. Applications

Let P_1 and P_2 be the properties of *f.p.* groups defined in Section 2.

Corollary 1:

(*i*), (*ii*) *the complements of* P_1 *and* P_2 *are Markov properties.*

Proof.

(*i*) Let U_1 and U_2 be two nonisomorphic universal groups. (For their existence cf. the comment at the end of Section 1.) Then there are two isomorphisms f and g, such that $f(U_1)=H_1$, $H_1 \mid< U_2$, and $g(U_2)=H_2$, $H_2 \mid< U_1$. Consequently, $g(f(U_1)) \mid< U_1$. Hence, every universal *f.p.* group is isomorphic to a proper subgroup of itself, and by the Theorem, the property P_1 is the complement of a Markov property.

(*ii*) Similarly, every *f.p.* factor-group of a given universal *f.p.* group U is isomorphic to a subgroup of U; i.e. every universal *f.p.* group enjoys P_2, which completes the proof of Corollary 1.

Corollary 2. *Let P be a union of (at most enumerably many) Markov properties. Then P is also a Markov property.*

Proof.

Let I be a (at most enumerable) set of indices, and let P_i, $i\in I$, denote a class of Markov properties. If $P=\bigcup_{i\in I} P_i$, then

$$P(G)\Leftrightarrow(\exists i\in I)P_i(G). \tag{3}$$

As P_i are Markov properties, by the Theorem one has

$$(\forall i\in I)(\forall U\in \bar{U})\neg P_i(U) \tag{4}$$

From (3) and (4) it follows that a group G enjoying P cannot be a universal group. Making use of the Theorem again, P is a Markov property.

Note that the property Q of "not being a universal *f.p.* group" is a Markov property. Furthermore, Q is the maximal Markov property, in the sense that Q contains any other such property. Also, it immediately follows that it is not algorithmically recognizable whether an *f.p.* group is universal or not.

We can conclude that the given test may be very efficient, which is well illustrated by simplicity of the above proofs.

5. Further remarks

1. Markov properties of groups were in fact defined in analogy to Markov properties of semigroups. The latter properties are also algorithmically unrecognizable [5]. The existence of a universal *f.p.* semigroup was proved in [7]. Hence, a theorem completely analogous to that of Section 3 (with analogous corollaries) can be stated for semigroups.

2. Simplification obtained by this new look on Markov properties directs one to search for possibly recursively recognizable algebraic properties only among those particular ones which contain some but not all of the universal *f.p.* groups. However, even for some of the properties from the latter class, one can immediately conclude that they are not recognizable. This applies e.g. to those of them which are strongly Markov properties [3]. (Unfortunately, an analogue of the Theorem of Section 3 cannot be stated for strongly Markov properties, since there exists no universal *f.p.* group with solvable word problem for the class of all *f.p.* groups with solvable word problem [6].)
The above consideration strongly suggests the following

Conjecture: *Every nontrivial algebraic property of f.p. groups is recursively unrecognizable.*
Further research in this direction is in progress.

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