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ON THE PLANE CREEPING FLOW OF SECOND ORDER FLUIDS WITH MIXED BOUNDARY CONDITIONS

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Teodor M. Atanacković

Summary

In this work we show when does equation of motion for slow flow of a second order Rivlin-Ericksen fluid reduce to that of a Newtonian fluid, when mixed no-slip, no-shear boundary conditions are imposed. Also, stability analysis for the rest state of Newtonian and second order fluids with mixed boundary conditions is given.

1. Introduction

Second-order Rivlin-Ericksen fluids are simple fluids for which the stress tensor depends only on the first and second Rivlin-Ericksen tensors [10]. Due to the relative simplicity that the equation of motion have, for second order fluids, they have been subject of many investigations. In [10] some of the properties of the flows of second order fluids are given. An important property, proved by Tanner [9] is that the velocity field for slow (creeping) flow of second--order fluids is the same as the velocity field for creeping flow of Newtonian fluid (Stokes flow) if the velocity boundary conditions are imposed. Recently Huilgol [4] estabilished uniqueness Theorem for such flows, i. e., he gave conditions under which the Stokes velocity field is the only velocity field for creeping flows of second order fluids. Using the method of Huilgol we will prove a result equivalent to the Tanner's, for creeping flow of second-order fluids with mixed, no-slip, no-shear boundary conditions. Our results will be valid for bounded flow domains. Extension to the unbounded domains could be done for certain flow geometries.

Stability analysis for flows fo second-order fluids was presented in [1], [2] and [5]. In our work we will analyse stability of the rest state of second-order fluids with mixed no-slip, no-shear boundary conditions. Even in this simple situation in which fluid fills a container and is in the state of rest, second-order Rivlin-Ericksen fluids show interesting property that they can be unstable, in certain cases, in the sense that small disturbances superimposed on the rest state, can grow up in time. This property of the second order fluids was noted earlier in [1] for no-slip boundary conditions. We state now our first result in the form of the following Theorem:

2. Theorem 2.1. *If $\psi \in C^5(\bar{\Omega})$ is stream function for creeping flow of a second-order fluid in a bounded domain Ω, with piecewise smooth boundary $\partial\Omega$ on which no-slip and no-shear boundary conditions are imposed, then the equations of motion reduce to*

$$\nabla^4 \psi = 0. \tag{2.1}$$

Remark 1. Equation (2.1) implies that second-order viscosities will come into the solution *only* through the boundary conditions.

Proof. Following Huilgol [4] the equations of motion for plane creeping flow of a second-order fluid can be written as:

$$w + \delta \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} w) = 0 \tag{2.2}$$

where $w = \nabla^4 \psi$, ψ is stream function, $\delta = \gamma/\nu$ is the ratio of viscosities and

$$\underset{\sim}{u} = \frac{\partial \psi}{\partial y} \underset{\sim}{i} - \frac{\partial \psi}{\partial x} \underset{\sim}{j} \tag{2.3}$$

is the velocity vector. Multiplying equation (2.2) by w and integrating over Ω we get

$$\int_{\Omega} w^2 da + \delta \int_{\Omega} w \underset{\sim}{u} \cdot \underset{\sim}{\nabla} w da = 0 \tag{2.4}$$

Now, since div $\underset{\sim}{u} = 0$ (continuity equation)

$$w \underset{\sim}{u} \cdot \underset{\sim}{\nabla} w = \operatorname{div}\left(\frac{w^2}{2} \underset{\sim}{u}\right) \tag{2.5}$$

so that

$$\int_{\Omega} w \underset{\sim}{u} \cdot \underset{\sim}{\nabla} w da = \frac{1}{2} \int_{\partial\Omega} w^2 \underset{\sim}{u} \cdot \underset{\sim}{n} ds \tag{2.6}$$

where we used the Divergence Theorem and where $\underset{\sim}{n}$ is the unit outward normal. Therefore equation (2.4) becomes

$$\int_{\Omega} w^2 da + \frac{\delta}{2} \int_{\partial\Omega} w^2 \underset{\sim}{u} \cdot \underset{\sim}{n} ds = 0 \tag{2.7}$$

Now $\gamma > 0$ and also $\nu < 0$ (cf. Coleman [1]) so that $\delta < 0$ Therefore $w \equiv 0$ will be solution of (2.2) if and only if

$$\int_{\partial\Omega} w^2 \underset{\sim}{u} \cdot \underset{\sim}{n} ds \leqslant 0. \tag{2.8}$$

To see that in the case of no-slip, no-shear conditions, (2.8) is satisfied, note that on the no slip part of $\partial\Omega$ $\underset{\sim}{u} = 0$, while on no-shear part of $\partial\Omega$ $\underset{\sim}{u} \cdot \underset{\sim}{n} = 0$, since $\partial\Omega$ is assumed independent of time. Therefore for the mixed no-slip, no--shear boundary conditions

$$\int_{\partial\Omega} w^2 \underset{\sim}{u} \cdot \underset{\sim}{n} = 0 \tag{2.9}$$

and the result of the Theorem follows.

Remark 2. We showed that the stream function for Stokes flow satisfies the same differential (biharmonic) equation as the stream function for creeping flow of second-order fluids, if mixed boundary conditions are imposed. The next proposition will give conditions under which the solution of the equation of motion (i. e., $\psi=\psi(x, y)$) for Stokes flow represents a solution of the equation of motion for creeping flow of second-order fluids.

Proposition 2.2. *Let Ω be as in Theorem 2.1, and suppose the no--shear boundary condition is imposed on the parts of $\partial\Omega$ that lie on a straight line. Then if $\psi(x, y)$ is a stream function for Stokes flow in Ω it is also a stream function for the creeping flow of second-order fluids in Ω.*

Proof. We may assume that the parts of $\partial\Omega$ on which the no shear stress condition is imposed lie on the x-axis of the coordinate system. By Theorem 2.1 both flows satisfy

$$\nabla^4\psi=0 \tag{2.10}$$

in Ω. Also, boundary conditions on the no-slip part are the same. On the no--shear part of $\partial\Omega$, Stokes flow satisfies

$$\frac{\partial u}{\partial y}=\psi_{yy}=0; \quad v=-\psi_x=0 \tag{2.11}$$

while creeping flow of second-order fluid satisfies

$$\psi_{yy}-\psi_{xx}+\gamma\{\psi_{xx}+\psi_{yy}\}\psi_{xy}=0. \tag{2.12}$$

Now, since from $(2.11)_2$ $\psi_x=0$ on parts of the x-axis, then on these parts $\psi_{xx}=0$ and therefore (2.12) is satisfied.

2.3 Example 1*. Next we give two examples illustrating use of the above results as well as a way of using them for unbounded domains. We consider first shear creeping flow of a second-order fluid over a flat plate with a transverse no shear slot. Stokes flow for this configuration has been solved by Philip [7].

Boundary conditions are:

$$\begin{array}{llll} \psi=0, & T_{xy}=0 & \text{on} & y=0,\ |x|<1 \\ \psi=0 & u=0 & \text{on} & y=0\ \ |x|\geqslant 1 \\ \lim\limits_{y\to\infty}\dfrac{\partial u}{\partial y}=C & & & \text{a constant} \end{array} \tag{2.13}$$

By the Proposition 2.2 ψ for Stokes flow (cf. Philip [7])

$$\psi=\frac{1}{2}Cy^2+C\frac{1}{4}Re\{[\sqrt{z^2-1}-z][\bar{z}-z]\} \tag{2.14}$$

is also a solution for second order fluids. In (2.14) the standard notation for complex-variable analysis is used.

*) Since in this and next example $\psi\notin C^5(\bar{\Omega})$, Theorem 2.1 does not hold, so that we do not claim that $\nabla^4\psi=0$ is the only solution.

2.4. Example 2. As a second example consider the stickslip flow. The solution for case of Newtonian fluids was given by Richardson [8], and we cite it in the Appendix.

Boundary conditions are

$$(2.15)\qquad \begin{array}{llll} \psi=1 & T_{xy}=0 & \text{for} & x\geqslant 0 \\ & u=0 & \text{for} & x<0 \\ \psi=-1 & T_{xy}=0 & \text{for} & x\geqslant 0 \\ & u=0 & \text{for} & x<0. \end{array}$$

Since the part of the boundary where the no-shear condition is applied is a straight line, Proposition 2.2 applies and the velocity distribution for Newtonian flow satisfies the equations of motion for flow of second-order fluids.

3. *Stability of the rest state of Newtonian and fluids of second order in bounded domains with mixed boundary conditions.*

The basic stability theorem for the rest state of simple fluids with *no slip* boundary conditions was proved by Joseph [5]. He found that Newtonian fluids in bounded domains with no-slip boundary conditions are stable in the state of rest, while second-order fluids under the same conditions are unstable, a fact earlier discovered by Coleman et al [1], Gubta [3] and Craik [2]. We will show that the same is true if no-slip boundary conditions are replaced by mixed, no-slip no-shear boundary conditions, when the no-shear condition is imposed on the segment of a straight line.

First, we will write the equations of motion for the perturbed flow. Suppose a fluid at rest occupies a domain Ω with a piecewise smooth boundary $\partial\Omega$. Suppose further that on the part $\partial\Omega_1$, of the boundary the no-slip condition is imposed while on the part $\partial\Omega_2$ the no-shear condition is imposed. We assume that the disturbed velocity field is of the form

$$(3.1)\qquad \underset{\sim}{v}=\varepsilon \underset{\sim}{u}(x, y)\, e^{-\sigma t}$$

where t is time and $\varepsilon \ll 1$. The rest state is called stable if the equations of motion together with the boundary conditions imply $Re\,\sigma<0$. Substituting (3.1) into the equation of motion we get

$$(3.2)\qquad \sigma \underset{\sim}{u}+\underset{\sim}{\nabla} p=K(\sigma)\nabla^2 \underset{\sim}{u}$$

where p is the pressure in the disturbed flow. We also neglected higher order therms in ε. The function $K(\sigma)$ is given as

$$(3.3)\qquad K(\sigma)=\begin{cases} \mu & \text{for Newtonian fluids} \\ \mu-\nu\sigma & \text{for 2nd order fluids} \end{cases}$$

Introducing the stream function and eliminating p by cross differentiation we get

$$(3.4)\qquad \nabla^4\psi+\frac{\sigma}{K(\sigma)}\nabla^2\psi=0$$

We impose the following boundary conditions

$$(3.5)\qquad \begin{aligned} \frac{\partial \psi}{\partial n} &= 0 \quad &&\text{on} \quad \partial\Omega_1 \\ \nabla^2 \psi &= 0 \quad &&\text{on} \quad \partial\Omega_2 \\ \psi &= 0 \quad &&\text{on} \quad \partial\Omega_1 \cup \partial\Omega_2. \end{aligned}$$

Equation (3.4) together with (3.5) defines a spectral problem. We can rewrite it as:

$$(3.6)\qquad A\psi - \Lambda B\psi = 0$$

where $A = \nabla^4(\cdot)$, and $B = -\nabla^2(\cdot)$, with boundary conditions (3.5).

We prove now the following

Proposition 3.1. *There are countably many eigenvalues of the spectral problem* (3.6), (3.5). *They are all real and positive, and the only accumulation point is at infinity.*

Proof. It can be easily shown that both operators A and B are positive definite. Then, if $\hat{\psi}$ is an eigenvector that corresponds to the eigenvalue $\hat{\Lambda}$ we have

$$(3.7)\qquad \hat{\Lambda} = \frac{(A\hat{\psi}, \hat{\psi})}{(B\hat{\psi}, \hat{\psi})} > 0$$

since both A and B are positive. Other properties asserted by the proposition follow directly as spectral properties of positive definite operators and their proof is given, for example, in [6] page 37. If follows, then, that we may form an increasing sequence of positive numbers from the set of eigenvalues of (3.6) and (3.5), i. e.,

$$(3.8)\qquad 0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_n < \cdots$$

To apply results of Proposition 3.1 to our stability problem, suppose that $\partial\Omega_2$ is part of a straight line. Then, condition $\nabla^2\psi = 0$, $\psi = 0$ on $\partial\Omega_2$ equivalent to the boundary condition $T_{nt} = 0$, i. e., shear stress is zero, for both Newtonian and second-order fluids. Therefore using $(3.3)_1$ and (3.4) we have

$$(3.9)\qquad \frac{\sigma_n}{\mu} = \Lambda_n \quad \text{or} \quad \sigma_n = \mu\Lambda_n > 0$$

since $\mu > 0$ and $\Lambda_n > 0$. We state the last result as

Theorem 3.2. *Newtonian fluids subjected to no-slip, no-shear boundary conditions are stable in the state of rest if the no-slip boundary is part of a straight line.*

For second-order fluids we have from $(3.3)_2$ *and* (3.4)

$$(3.10)\qquad \frac{\sigma_n}{\mu - \nu\sigma_n} = \Lambda_n$$

or

$$(3.11)\qquad \sigma_n = \frac{\Lambda_n \mu}{1 + \nu\Lambda_n}$$

Now, since $\mu>0$, $\Lambda_n>0$ *and* $\nu<0$ (*cf.* [1]) *if follows that there is an* N *such that*

(3.12) $$\sigma_n<0 \quad \textit{for all } n>N.$$

We state this as

Theorem 3.3. *Rivlin-Ericksen fluids of the second order with* $\nu<0$ *subject to the no-shear, no-slip boundary conditions are unstable in the state of rest if the no-slip boundary condition is part of a straight line.*

Appendix

Flow field for stick-slip problem

Here we give result for stick-slip flow from the work of Richardson [8]. If we denote

(A. 1) $$u_0=\frac{\partial\psi_0}{\partial y}; \qquad v_0=-\frac{\partial\psi_0}{\partial x}$$

then for $x>0$, the stream function ψ_0 is given as

(A. 2) $$\psi_0=y-\frac{3}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{B_-(-in\pi)}xe^{-n\pi x}\sin n\pi y$$
$$+\frac{3}{2}\sum_{n=1}^{\infty}(-1)^n\lim_{w\to -in\pi}\left[\frac{d}{dw}\left\{\frac{i}{w^2B_-(w)}\right\}\right]e^{-n\pi x}\sin n\pi y$$

while for $x\leqslant 0$ one has

(A. 3) $$\psi_0=\frac{1}{2}y(3-y^2)-3\sum_{n=1}^{\infty}Re\left\{\frac{B+\left(\frac{1}{2}\alpha_n\right)}{\sin h^2\frac{1}{2}\alpha_n}\left[\frac{\sin h\frac{1}{2}\alpha_n y}{\sin h\frac{1}{2}\alpha_n}-\frac{y\cos h\frac{1}{2}\alpha_n y}{\cos h\frac{1}{2}\alpha_n}\right]e^{-i\frac{1}{2}\alpha_n x}\right\}.$$

In above equations the following was used:

(A. 4) $$B_+=-\frac{1}{3}\prod_{n=1}^{\infty}\frac{\left(1+\frac{2w}{\alpha_n}\right)\left(1-\frac{2w}{\alpha_n}\right)}{\left(1+\frac{w}{in\pi}\right)^2}$$
$$B_-(w)=-3B_+(w)$$

where the α_n's are the non-zero roots of the equation $\alpha=\sin h\,\alpha$ in the first quadrant, i. e., $0<\arg\alpha_n<\frac{1}{2}\pi$.

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Faculty of Technical Sciences
University of Novi Sad
21000 Novi Sad, Yugoslavia