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HEAT CONDUCTION AND H—FUNCTION

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Abstract

The object of this paper is to evaluate the integrals involving the product of H-function, generalized hypergeometric function of two variables and Hermite polynomials with the help of the finite difference operator E. This integral has been used to obtain a solution of a problem of heat conduction and also an expansion formula for the product of H-function and generalized hypergeometric function of two variables.

1. Introduction

Recently, Fox [6, p. 408] has introduced the H-function in the form of Mellin-Barnes type integral as

$$(1.1)\qquad H_{p,q}^{n,l}[x]=H_{p,q}^{n,l}\left[x\,\middle|\,\begin{matrix}\{(a_p,\ e_p)\}\\ \{(b_q,\ f_q)\}\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{n}\Gamma(b_j-f_js)\prod_{j=1}^{l}\Gamma(1-a_j+e_js)}{\prod_{j=n+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=l+1}^{p}\Gamma(a_j-e_js)}\,x^s\,ds,$$

where $\{(a_p,\ e_p)\}$ represents the set of parameters $(a_1,\ e_1),\ldots,(a_p,\ e_p)$, x is not equal to zero and an empty product is interpreted as unity; $p,\ q,\ n$ and l are integers satisfying $0\leqslant n\leqslant q$, $0\leqslant l\leqslant p$; $e_j\,(j=1,\ldots,p)$, $f_h\,(h=1,\ldots,q)$ are positive numbers and $a_j(j=1,\ldots,p)$, $b_h(h=1,\ldots,q)$, are complex numbers. L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j-f_js)$ $(j=1,\ 2,\ldots,n)$ lie to the right and those of $\Gamma(1-a_j+e_js)$ $(j=1,\ 2,\ldots,l)$ to the left of L. These assumptions for the H-function will be adhered to throughout this paper.

In our present work, we shall require the following results [7, p. 33 with $\omega=1$],

$$(1.2)\qquad E_\alpha f(\alpha)=f(\alpha+1)$$

and [2, p. 2, (2. 1)]

(1.3)
$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2\nu}(x) H_{p,q}^{n,l}\left[zx^{-2m} \middle| \begin{matrix} \{(a_p, e_p)\} \\ \{(b_q, f_q)\} \end{matrix} \right] dx$$
$$= \sqrt{\pi}\, 2^{2(\nu-\rho)} H_{p+1,q+1}^{n+1,l}\left[z\, 2^{2m} \middle| \begin{matrix} \{(a_p, e_p)\}, (1+\rho-\nu, m) \\ (1+2\rho, 2m), \{(b_q, f_q)\} \end{matrix} \right]$$

where m is a positive number, $\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \leqslant 0$, $\sum_{j=1}^{l} e_j - \sum_{j=l+1}^{p} e_j + \sum_{j=1}^{n} f_j - \sum_{j=n+1}^{q} f_j$

$= M > 0$ and $|\arg z| < \frac{1}{2} \prod M$ and $\rho = 0, 1, 2, \ldots$.

Appell and Kampé de Fériet [1] have defined a generalized hypergeometric function of two variables as

(1.4)
$$F\left[\begin{matrix} \lambda' \\ \mu' \\ \nu' \\ \sigma' \end{matrix} \middle| \begin{matrix} \alpha_1, \ldots, \alpha_{\lambda'} \\ \beta_1 \beta_1', \ldots, \beta_{\mu'}, \beta_{\mu'}' \\ \gamma_1, \ldots, \gamma_{\nu'} \\ \delta_1, \delta_1^1, \ldots, \delta_{\sigma'}, \delta_{\sigma'}' \end{matrix} \middle| x, z \right] = \sum_{m,n=0}^{\infty} \frac{\prod_{i=1}^{\lambda'} (\alpha_i)_{m+n} \prod_{i=1}^{\mu'} (\beta_i)_m (\beta_i')_n x^m z^n}{\prod_{i=1}^{\nu'} (\gamma_i)_{m+n} \prod_{i=1}^{\sigma'} (\delta_i)_m (\delta_i')_n m!\, n!}$$

where (α_p), $(\alpha_p)_{m+n}$ and $(\alpha)_r$ stand for $\alpha_1, \ldots, \alpha_p$; $(\alpha_1)_{m+n}, \ldots, (\alpha_p)_{m+n}$ and $\Gamma(\alpha+r)/\Gamma(\alpha)$ respectively. The series given by (1.4) is absolutely convergent when $\lambda' + \mu' \leqslant \nu' + \sigma' + 1$.

Also for special values of λ', μ', ν' and σ' the function defined above degenerates into the double hypergeometric function [5] namely F_1, F_2, F_3, F_4, Φ_2 and ψ_2.

In this paper we have evaluated an integral involving the product of H-function, generalized hypergeometric function of two variables and Hermite polynomial. This integral has also been used to obtain the solution of a problem of heat conduction given by Bhonsle [3].

2. Integral

On multiplying (1.3) by
$$\frac{\prod_{i=1}^{\lambda'} \Gamma(\alpha_i+\alpha) \prod_{i=1}^{\mu'} \Gamma(\beta_i+\beta)\Gamma(\beta_i'+\beta')}{\prod_{i=1}^{\nu'} \Gamma(\gamma_i+\alpha) \prod_{i=1}^{\sigma'} \Gamma(\delta_i+\beta)\Gamma(\delta_i'+\beta')} u^{\beta} v^{\beta'}$$

and applying the operator exp. $\{E_\alpha E_\beta E_\rho^d + E_\alpha E_{\beta'} E_\rho^d\}$,
we obtain

$$\sum_{r,s=0}^{\infty} \int_{-\infty}^{\infty} x^{2(\rho+rd+sd)} e^{-x^2} \frac{\prod_{i=1}^{\lambda'} \Gamma(\alpha_i+\alpha+r+s) \prod_{i=1}^{\mu'} \Gamma(\beta_i+\beta+r)\Gamma(\beta_i'+\beta'+s)}{\prod_{i=1}^{\nu'} \Gamma(\gamma_i+\alpha+r+s) \prod_{i=1}^{\sigma'} \Gamma(\delta_i+\beta+r)\Gamma(\delta_i'+\beta'+s)}$$

$$H_{2\nu}(x)\,H_{p,q}^{n,l}\left[zx^{-2m}\left|\begin{matrix}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{matrix}\right.\right]\frac{u^{\beta+r}v^{\beta'+r}}{r!\,s!}\,dx$$

$$=\sqrt{\pi}\sum_{r,s=0}^{\infty}2^{2(\upsilon-\rho-rd-sd)}\frac{\prod_{i=1}^{\lambda'}\Gamma(\alpha_i+\alpha+r+s)\prod_{i=1}^{\mu'}\Gamma(\beta_i+\beta+r)\,\Gamma(\beta_i'+\beta'+s)}{\prod_{i=1}^{\nu'}\Gamma(\gamma_i+\alpha+r+s)\prod_{i=1}^{\sigma'}\Gamma(\delta_i+\beta+r)\,\Gamma(\delta_i'+\beta'+s)}$$

$$H_{p+1,q+1}^{n+1,l}\left[z\,2^{2m}\left|\begin{matrix}\{(a_p,e_p)\},\ (1+\rho-\nu+rd+sd,\ m)\\ (1+2\rho+2rd+2sd,\ 2m),\ \{(bq,fq)\}\end{matrix}\right.\right]\frac{u^{\beta+r}v^{\beta'+s}}{r!\,s!}$$

which gives

$$(2.1)\qquad \int_{-\infty}^{\infty}x^{2\rho}e^{-x^2}F\left[\begin{matrix}\lambda'\\ \mu'\\ \nu'\\ \sigma'\end{matrix}\left|\begin{matrix}\alpha_1,\ldots,\alpha_{\lambda'}\\ \beta_1,\beta_1',\ldots,\beta_{\mu'},\beta_{\mu'}'\\ \gamma_1,\gamma_2,\ldots,\gamma_{\nu'}\\ \delta_1,\delta_1',\ldots,\delta_{\sigma'},\delta_{\sigma'}'\end{matrix}\right|ux^{2d},vx^{2d}\right]H_{2\nu}(x)$$

$$H_{p,q}^{n,l}\left[zx^{-2m}\left|\begin{matrix}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{matrix}\right.\right]dx$$

$$=4^{\nu-\rho}\sqrt{\pi}\sum_{r,s=0}^{\infty}\frac{\prod_{i=1}^{\lambda'}(\alpha_i)_{r+s}\prod_{i=1}^{\mu'}(\beta_i)_r(\beta_i')_s}{\prod_{i=1}^{\nu'}(\gamma_i)_{r+s}\prod_{i=1}^{\sigma'}(\delta_i)_r(\delta_i')_s}\frac{\left(\frac{u}{4^d}\right)^r\left(\frac{v}{4^d}\right)^s}{r!\,s!}$$

$$H_{p+1,q+1}^{n+1,l}\left[z\,2^{2m}\left|\begin{matrix}\{(a_p,e_p)\},\ (1+\rho-\nu+rd+sd,\ m)\\ (1+2\rho+2rd+2sd,\ 2m),\ \{(b_q),f_q)\}\end{matrix}\right.\right].$$

The change of order of summation and integration is permissible here under the conditions given in (1.3) and (1.4).

3. Heat Conduction

Recently Bhonsle [3] has given the solution of the partial differential equation

$$(3.1)\qquad \frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}-kux^2$$

where $u(x,t)$ tends to zero for large values of t and when $|x|\to\infty$, as

$$(3.2)\qquad u(x,t)=\sum_{\mu=0}^{\infty}A_\mu e^{-(1+2\mu)kt-\frac{x^2}{2}}H_\mu(x).$$

The equation (3.1) can be associated with a heat conduction equation [4, p. 130]:

$$(3.3)\qquad \frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}-h(u-u_0)$$

provided that $u_0=0$ and $h=kx^2$.

When $t=0$, let

$$u(x,0)=x^{2\rho}e^{-x^2}F\left[\begin{array}{c|l|c} \lambda' & \alpha_1,\ldots,\alpha_{\lambda'} & \\ \mu' & \beta_1,\beta_1',\ldots,\beta_{\mu'},\beta_{\mu'}' & \\ \nu' & \gamma_1,\gamma_2,\ldots,\gamma_{\mu'} & ux^{2d}\ \ vx^{2d} \\ \sigma' & \delta_1,\delta_1',\ldots,\delta_{\sigma'},\delta_{\sigma'}' & \end{array}\right]$$

$$H_{p,q}^{n,l}\left[zx^{-2m}\left|\begin{array}{l}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{array}\right.\right]$$

then

$$(3.4)\quad x^{2\rho}e^{-x^2}F\left[\begin{array}{c|l|c} \lambda' & \alpha_1,\ldots,\alpha_{\lambda'} & \\ \mu' & \beta_1,\beta_1',\ldots,\beta_{\mu'},\beta_{\mu'}' & \\ \nu' & \gamma_1,\ldots,\gamma_{\nu'} & ux^{2d},vx^{2d} \\ \sigma' & \delta_1,\delta_1',\ldots,\delta_{\sigma'},\delta_{\sigma'}' & \end{array}\right]H_{p,q}^{n,l}\left[zx^{-2m}\left|\begin{array}{l}\{a_p,e_p)\}\\ \{(b_q,f_q)\}\end{array}\right.\right]$$

$$=\sum_{\mu=0}^{\infty}A_\mu e^{-x^2/2}H_\mu(x).$$

Now multiplying both sides of (3.4) by $H_\lambda(x)$ and integrating from $-\infty$ to ∞ with respect to x and making use of the orthogonality property for the Hermite polynomials [5, p. 289, (9) and (11)]. we get

$$(3.5)\qquad A_\lambda=\frac{2^{\lambda-2\rho-\frac{1}{2}}}{\lambda!}\sum_{r,s=0}^{\infty}\frac{\prod_{i=1}^{\lambda'}(\alpha_i)_{r+s}\prod_{i=1}^{\mu'}(\beta_i)_r(\beta_i')_s}{\prod_{i=1}^{\nu'}(\gamma_i)_{r+s}\prod_{i=1}^{\sigma'}(\delta_i)_r(\delta_i')_s}\frac{\left(\frac{u}{4^d}\right)^r\left(\frac{v}{4^d}\right)^s}{r!\,s!}$$

$$H_{p+1,q+1}^{n+1,l}\left[z\,2^{2m}\left|\begin{array}{l}\{(a_p,e_p)\},\left(1+\rho+rd+sd-\frac{\lambda}{2},m\right)\\ (1+2\rho+2rd+2sd,2m),\{b_q,f_q)\}\end{array}\right.\right].$$

With the help of (3.5), the solution (3.2) becomes

$$(3.6)\qquad u(x,t)=\sum_{\mu,r,s=0}^{\infty}\frac{2^{\mu-2\rho-\frac{1}{2}}\prod_{i=1}^{\lambda'}(\alpha_i)_{r+s}\prod_{i=1}^{\mu'}(\beta_i)_r(\beta_i')_s}{\prod_{i=1}^{\nu'}(\gamma_i)_{r+s}\prod_{i=1}^{\sigma'}(\delta_i)_r(\delta_i')_s}\frac{\left(\frac{u}{4^d}\right)^r\left(\frac{v}{4^d}\right)^s}{r!\,s!\,\mu!}$$

$$e^{-(1+2\mu)kt-\frac{x^2}{2}}H_\mu(x)\,H_{p+1,q+1}^{n+1,l}\left[z\,2^{2m}\left|\begin{array}{l}\{(a_p,e_p)\},\left(1+\rho+rd+sd-\frac{\mu}{2},m\right)\\ (1+2\rho+2rd+2sd,2m),\{(b_q,f_q)\}\end{array}\right.\right],$$

where conditions of validity being the same as given in (1.3) and (1.4).

4. Expansion

From (3.4) and (3.5), we have

$$(4.1)\quad x^{2\rho} e^{-x^2} F\left[\begin{array}{l|l|l} \lambda' & \alpha_1, \ldots, \alpha_{\lambda'} & \\ \mu' & \beta_1, \beta_1', \ldots, \beta_{\mu'}, \beta_{\mu'}' & \\ \nu' & \gamma_1, \ldots, \gamma_{\nu'} & ux^{2d}, vx^{2d} \\ \sigma' & \delta_1, \delta_1', \ldots, \delta_{\sigma'}, \delta_{\sigma'}' & \end{array}\right] H_{p,q}^{n,l}\left[zx^{-2m}\left|\begin{array}{l}\{(a_p, e_p)\}\\ \{(b_q, f_q)\}\end{array}\right.\right]$$

$$= \sum_{\mu, r, s=0}^{\infty} \frac{2^{\mu-2\rho-\frac{1}{2}} \prod_{i=1}^{\lambda'} (\alpha_i)_{r+s} \prod_{i=1}^{\mu'} (\beta_i)_r (\beta_i')_s}{\prod_{i=1}^{\nu'} (\gamma_i)_{r+s} \prod_{i=1}^{\sigma'} (\delta_i)_r (\delta_i')_s} \frac{\left(\frac{u}{4^d}\right)^r \left(\frac{v}{4^d}\right)^s}{\mu!\, r!\, s!} e^{-\frac{x^2}{2}} H_\mu(x)$$

$$H_{p+1,q+1}^{n+1,l}\left[z\, 2^{2m}\left|\begin{array}{l}\{(a_p, e_p)\}, \left(1+\rho+rd+sd-\frac{\mu}{2}, m\right)\\ (1+2\rho+2rd+2sd,\ 2m)\{b_q, f_q)\}\end{array}\right.\right],$$

above is the expansion formula for the product of *H*-function and generalized hypergeometric function of two variables; the conditions of validity being the same as stated in (1.3) and (1.4).

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