PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 21 (35), 1977, pp. 185—196

APPROXIMATE SOLUTIONS OF THE OPERATOR LINEAR DIFFERENTIAL EQUATION I

B. Stanković

(Received February 28, 1976)

From all the differential equations for Mikusiński operators the differential equation:

$$\sum_{k=0}^{n} a_k(\mathbf{s})\, \mathbf{x}^{(k)}(\lambda)=0, \tag{1}$$

where $a_k(\mathbf{s})$ are polynomials in the differential operator $\mathbf{s}$ with numerical coefficients:

$$a_k(\mathbf{s})=\sum_{\nu=0}^{d_k} \alpha_{k,\nu}\, \mathbf{s}^{\nu}, \qquad d_k \leqslant m, \qquad k=0,\ 1, \ldots, n \tag{2}$$

is of the biggest interest to applications. The numerical linear partial differential equations with constant coefficients reduce to them. Our aim is to construct for equation (1) simple and good studied operators which approximate, in a special sense, the solutions of equation (1). The needed calculations are accommodated to the use of a computer. Finally we shall apply these results to the partial differential equations for numerical functions.

To make a unity of this material we shall briefly repeat some known notions and results (see [4]) and give also some new ones. Our results are not given in a form: theorem-proof. Because here are important not only single results but the access and idea. The main result is that the approximate solutions, if the solutions are from $\mathcal{C}$, $\mathcal{L}$ or $\mathcal{D}'$, are expressed by a unique class of functions (the class of Wright's functions).

1. The field of Mikusiński operators

The ring $\mathcal{C}$ is the ring of continuous complex valued functions defined over $[0, \infty)$ with the operations: sum and finite convolution:

$$f * g=\{\int_0^t f(t-u)\, g(u)\, du\}.$$

We denote by $\mathbf{f}$ or $\{f(t)\}$ the representation of $f(t) \in \mathcal{C}_{[0, \infty)}$ in $\mathcal{C}$.

The ring $\mathcal{L}$ is the ring of local integrable functions over $[0, \infty)$ with the same operations as in $\mathcal{C}$. The quotient field of these rings is the field $\mathcal{M}$ of Mikusiński operators. The both rings $\mathcal{C}$ and $\mathcal{L}$ have not the unit element. The field $\mathcal{M}$ is not algebraically closed.

1.1 *Binary relations and the operator of absolute value in* $\mathcal{L}$

In $\mathcal{L}$ we shall define two binary relations:

$$\mathbf{f} \leqslant \mathbf{g} \Leftrightarrow f(t) \leqslant g(t), \quad t \in [0, \infty)$$

$$\mathbf{f} \leqslant_T \mathbf{g} \Leftrightarrow f(t) \leqslant g(t), \quad t \in [0, T].$$

The first one is an ordering relation.

By $|\mathbf{f}|$ we denote the mapping $\mathcal{L}$ into $\mathcal{L}: \mathbf{f} \to |\mathbf{f}| = \{|f(t)|\}$ which be named the operator of absolute value. It has the following properties which we need:

1. $|\mathbf{f}+\mathbf{g}| \leqslant |\mathbf{f}| + |\mathbf{g}|$;

2. $|\alpha \mathbf{f}| = |\alpha| |\mathbf{f}|$, α complex number;

3. $|\mathbf{fg}| = \{|\int_0^t f(t-u) g(u) du|\} \leqslant |\mathbf{f}| |\mathbf{g}|$;

4. If for a fixed $\mathbf{f}$ there exists $M(T, f) = \sup_{0 \leqslant t \leqslant T} |f(t)|$,

then $|f| \leqslant_T M(T, f) \, l$, where $l = \{1\}$;

5. For a complex number α:

$$|(\alpha + \mathbf{f})^n \mathbf{g}| \leqslant \sum_{k=0}^{n} \binom{n}{k} |\alpha|^{n-k} |\mathbf{f}|^k |\mathbf{g}| \leqslant (|\alpha| + |\mathbf{f}|)^n |\mathbf{g}|$$

6. $l^p = \left\{\frac{t^{p-1}}{\Gamma(p)}\right\} \leqslant_T \frac{T^{p-1}}{\Gamma(p)} l, \quad p \geqslant 1$

1.2. *Convergence and approximation in* $\mathcal{M}$

The sequence $\{\mathbf{a}_n\} \subset \mathcal{M}$ converges to $\mathbf{a} \in \mathcal{M}$ if and only if there exists an element $\mathbf{q} \in \mathcal{M}$ such that $\{\mathbf{qa}_n\} \subset \mathcal{C}$ and for every $T < \infty$ $\{\mathbf{qa}_n\}$ converges uniformly in $[0, T]$ to $\mathbf{qa}$.

The defined convergence class is not topological and the sequential closure of a subset $S \subset \mathcal{M}$ does not always satisfy the condition $\overline{\overline{S}} = \overline{S}$ ($\overline{S}$ is the closure of S).

D e f i n i t i o n. The operator $\mathbf{a}$ approximates the operator $\mathbf{b}$ with a factor $\mathbf{q} \in \mathcal{M}$ and a measure ε if $\mathbf{q}^{-1}(\mathbf{a}-\mathbf{b}) \in \mathcal{L}$ and $|\mathbf{q}^{-1}(\mathbf{a}-\mathbf{b})| \leqslant \varepsilon \, l$; the operator $\mathbf{a}$ approximates locally the operator $\mathbf{b}$ with a factor $\mathbf{q} \in \mathcal{M}$ and a measure $\varepsilon(T)$ if $\mathbf{q}^{-1}(\mathbf{a}-\mathbf{b}) \in \mathcal{L}$ and $|\mathbf{q}^{-1}(\mathbf{a}-\mathbf{b})| \leqslant_T \varepsilon(T) \, l$.

If $\mathbf{q} = \mathbf{I}$, our definition gives the classical approximation in $\mathcal{L}$. If there exists $M(T, f)$ and $|\mathbf{f}^{-1}(\mathbf{a}-\mathbf{b})| \leqslant_T \varepsilon(T) \, l$, then $|\mathbf{a}-\mathbf{b}| \leqslant_T M(T, f) \, \varepsilon(T) \, l^2$.

1.3. *Mappings of an interval into $\mathcal{M}$*

Let Λ be an interval in $\mathcal{R}$ or a domain in $\mathcal{C}$, then $\mathbf{a}(\lambda)$ means mapping which maps Λ into $\mathcal{M}$. We say that $\mathbf{a}(\lambda)$ has a derivative $\mathbf{a}'(\lambda)$ in Λ if there exists $\mathbf{q}\in\mathcal{M}$ such that $\mathbf{qa}(\lambda)=\{b(\lambda, t)\}$, where $b(\lambda, t)$ is a numerical function of two variables which has the continuous partial derivative $b_\lambda'(\lambda, t)$ over $\Lambda\times\mathcal{R}^+$; then by definition $\mathbf{a}'(\lambda)=\mathbf{q}^{-1}\{b'_\lambda(\lambda, t)\}$.

2. Existence and the construction of the solution of equation (1)

The linearly independent solutions of the homogeneous equation (1) are of the form:

(3) $$\mathbf{x}(\lambda)=\lambda^i e^{\lambda\mathbf{w}}, \quad i=0,1, \dots, k-1,$$

where $\mathbf{w}$ is a k-tiple zero of the polynomial:

(4) $$F(\mathbf{w})=\sum_{k=0}^{n} a_k(\mathbf{s})\,\mathbf{w}^k, \quad a_k(\mathbf{s})=\sum_{\nu=0}^{d_k}\alpha_{k,\nu}\mathbf{s}^\nu, \quad d_k\leqslant m$$

(see [4] pp. 269—272 and 481).

The analysis and the construction of the solution of equation (1) require:

— To find the zeros of the polynomial $F(\mathbf{w})$.

— To establish the existence of the exponential operators (3) for the found zero $\mathbf{w}$.

— If the exponential operator (3) exists, it remains to analyse its character: is it an element of $\mathcal{C}$, $\mathcal{L}$, a distribution or only an operator.

2.1. *Zeros of the polynomial $F(w)$*

$F(\mathbf{w})$ is in reality a polynomial in $\mathbf{w}$ and $\mathbf{s}$. The equation

(5) $$F(\mathbf{s}, \mathbf{w})=\sum_{k=0}^{n}\sum_{\nu=0}^{d_k}\alpha_{k,\nu}\mathbf{s}^\nu\mathbf{w}^k=0$$

gives the zeros of the polynomial $F(\mathbf{w})$ as a function of $\mathbf{s}$. To find these solutions of equation (5) we shall use the theory of algebraic function ([1] pp. 153—213).

Let us multiply relation (5) by the integral operator to the power $m\geqslant\max d_k$, $0\leqslant k\leqslant n$:

(6) $$\sum_{k=0}^{n}\sum_{\nu=0}^{d_k}\alpha_{k,\nu}\,l^{m-\nu}\,\mathbf{w}^k=0$$

We shall consider now the correspondent polynomial in z and w; z and w are complex numbers:

(7) $$F(z, w)=\sum_{k=0}^{n}\sum_{\nu=0}^{d_k}\alpha_{k,\nu}\,z^{m-\nu}\,w^k=0$$

As the ring of convergent numerical power series $\sum_{i\geq 0} a_i z^i$ is isomorphic to the ring of operator series $\sum_{i\geq 0} a_i l^i$, it follows from the theory of analytical functions that the solutions of the equations (6) are of the form:

$$\mathbf{w}=l^{-\frac{q}{p}}\sum_{i\geq 0} a_i l^{\frac{i}{p}}, \qquad q, p\in N, \tag{8}$$

where $q=0$ if $d_n=m$ and if $d_n<m$ q can differ from zero.

To prove that the series (8) converges in $\mathcal{M}$, it is enough to use the fact that the correspodent numerical series $\sum_{i\geq 0} a_i z^{i/p}$ converges for at least one value $z_0\neq 0$ Then for every $i\geq i_0$

$$|a_i|\leq C/|z_0|^{\frac{i}{p}} \text{ and } |a_i l^{\frac{i}{p}}|\leq_T \frac{C}{T}(T/|z_0|)^{\frac{i}{p}}\frac{l}{\Gamma(i/p)},$$

which shows that the series (8) converges in $\mathcal{M}$.

2.2. *Existence of the exponential operator*

We know ([4] p. 442 and [5] p. 224) that $e^{\lambda \mathbf{w}}$, where $\mathbf{w}$ is given by series (8), is an exponential operator if $\frac{q}{p}<1$ and λ complex, or $\frac{q}{p}=1$ and λa_0 real. In the other cases, $\frac{q}{p}>1$, $e^{\lambda \mathbf{w}}$ does not exist.

2.3. *Character of the exponential operator*

Our exponential operator $e^{\lambda \mathbf{w}}$ where $\mathbf{w}$ is defined by relation (8) contains in reality operators of two forms:

$$\exp(-\mu\, l^{-\alpha}),\ \alpha>0 \text{ and } e^{\mathbf{f}},\ \mathbf{f}=\sum_{i\geq i_0} a_i l^{\frac{i}{p}-\alpha},\ \frac{i_0}{p}-\alpha>0.$$

The character of the exponential operator $\exp(-\mu\, l^{-\alpha})$, $0<\alpha<1$ is analysed in [6], [8] and [11]:

1. $0<\alpha<1$ and $|\arg\mu|<\frac{\pi}{2}(1-\alpha)$

$$\exp(-\mu\, l^{-\alpha})=\begin{Bmatrix} t^{-1}\Phi(0,-\alpha;\ -\mu t^{-\alpha}), & t\neq 0 \\ 0 & , t=0 \end{Bmatrix},$$

Φ is the function of E. M. Wright [12]. We see that our operator is from $\mathcal{C}$. For the special cases $\alpha=1/2$ and $\alpha=2/3$ see [10] pp. 115—116.

2. $0<\alpha<1$ and $|\arg \mu|=\frac{\pi}{2}(1-\alpha)$.

In this case

$$\exp(-\mu l^{-\alpha})=\mathbf{s}l^{1/2}\left\{t^{-1/2}\,\Phi\left(\frac{1}{2},\ -\alpha;\ -|\mu|\,e^{\pm\frac{\pi}{2}(1-\alpha)i}\,t^{-\alpha}\right)\right\}.$$

If $0<\alpha\leqslant\frac{2}{3}$, it is a distribution which is not a function, and for $\frac{2}{3}<\alpha<1$ it is a function which is not Lebesgue-integrable over $[0, T]$, $T>0$.

3. $0<\alpha<1$ and $\pi\geqslant|\arg\mu|>\frac{\pi}{2}(1-\alpha)$. The operator $\exp(-\mu l^{-\alpha})$ is an operator which is neither a distribution nor a function.

4. $\exp(-\mu l^{-1})=e^{-\mu s}$, $\mu>0$ is the translation operator.

5. $e^{\mathbf{f}}$, $\mathbf{f}\in\mathscr{L}$, then $(e^{\mathbf{f}}-\mathbf{I})\in\mathscr{L}$. In special case $\mathbf{f}=\mu l^{\alpha}$, $\alpha>0$, μ complex number, we have $\exp(\mu l^{\alpha})=\mathbf{I}+\{t^{-1}\,\Phi(0,\alpha;\ \mu t^{\alpha})\}$ where Φ is the function of E. M. Wright [13], [10]. If $0<\alpha<1$, $\exp(\mu l^{\alpha})-\mathbf{I}$ is from $\mathscr{L}$; if $\alpha\geqslant 1$, $\exp(\mu l^{\alpha})-\mathbf{I}$ is from $\mathscr{C}$. It is easy to prove this:

$$\begin{aligned}\exp(\mu l^{\alpha})&=\mathbf{I}+\mathbf{s}\sum_{k=1}^{\infty}\frac{1}{k!}\mu^k l^{\alpha k+1}\\&=\mathbf{I}+\mathbf{s}\left\{\sum_{k=1}^{\infty}\frac{\mu^k t^{\alpha k}}{\Gamma(k+1)\Gamma(\alpha k+1)}\right\}\\&=\mathbf{I}+\left\{\sum_{k=1}^{\infty}\frac{\mu^k t^{\alpha k-1}}{\Gamma(k+1)\Gamma(\alpha k)}\right\}\\&=\mathbf{I}+\{t^{-1}\,\Phi(0,\alpha;\ \mu t^{\alpha})\}\end{aligned}$$

We used here that $1/\Gamma(\alpha k)=0$ for $k=0$.

3. Numerical computations

3.1. *Calculation of p/q and a_0*

We saw that q/p and a_0 determine the existence of the equation (1). To find these two numbers we suppose that $\mathbf{w}$ is of the form (8) and that it satisfies equation (6). The least degree r of l has to appear at least twice, for $k=j$ and $k=i$:

$$r=m-d_j-\frac{q}{p}j=m-d_i-\frac{q}{p}i\leqslant m-d_k-\frac{q}{p}k,\qquad k=0,1,\ldots,n. \tag{9}$$

Whence

$$d_j+\frac{q}{p}j=d_i+\frac{q}{p}i\geqslant d_k+\frac{q}{p}k,\qquad k=0,1,\ldots,n. \tag{10}$$

The system (1) gives $\frac{q}{p}$ and is suitable to be treated by a computer. The coefficient of l^r gives a_0. This has to be a simple zero of the polynomial:

$$Q(a_0)=\alpha_{i,di}\,a_0^i+\alpha_{j,dj}\,a_0^j+\dots \tag{11}$$

and the polynomial (7) has to be irreducible.

3.2. *Computation of other coefficients* a_i

We suppose that the polynomial (7) is irreducible. Now we can introduce $\omega=l^{\frac{q}{p}}\mathbf{w}$, $l^{\frac{1}{p}}=\mathbf{u}$ in (6) and we shall obtain a polynomial in $\mathbf{u}$ and ω. It can be written in the form:

$$P(\mathbf{u},\omega)=\sum_{k=0}^{n}\sum_{\nu\geqslant 0}\frac{1}{k!\,\nu!}P_{u^\nu,\omega^k}(0,a_0)\,\mathbf{u}^\nu(\omega-a_0)^k$$

The coefficient of $\omega-a_0$ is $P_\omega(0,a_0)=Q'(a_0)\neq 0$.

Now we can write:

$$\omega-a_0=-\frac{1}{Q'(a_0)}\sum_{k=0}^{n}{\sum_{\nu\geqslant 0}}'\frac{1}{k!\,\nu!}P_{u^\nu,\omega^k}(0,a_0)\,\mathbf{u}^\nu(\omega-a_0)^k. \tag{12}$$

In this double sum we have to omit the couple of indexes (1,0). Using the relation $\omega-a_0=\sum_{i\geqslant 0}a_i\mathbf{u}^i$ we have all the coefficients from the relation (12):

$$a_1=-\frac{1}{Q'(a_0)}P_u(0,a_0)$$

$$a_2=-\frac{1}{Q'(a_0)}\left[\frac{1}{2}P_{u^2}(0,a_0)+\frac{1}{2}P_{\omega^2}(0,a_0)\,a_1^2+P_{u,\omega}(0,a_0)\,a_1\right]$$

4. Approximation of the solution of equation (1)

We saw that the linearly independent solutions of equation (1) are of the form (3) where $\mathbf{w}$ is given by the series (8). We shall give in this general access approximations just of the linearly independent solutions because they can be used for the construction of the solutions of the nonhomogeneous equation

$$\sum_{k=0}^{n}a_k(\mathbf{s})\,\mathbf{x}^{(k)}(\lambda)=\mathbf{f}(\lambda)$$

as for initial so for boundary value problem.

Let us suppose that we have computed the first i_0 coefficients a_i so that $i_0>q$. As the approximate solution of equation (1) we shall take:

$$\exp\left(\lambda \sum_{i=0}^{i_0} a_i\, l^{\frac{i-q}{p}}\right)$$

The difference from the exact solution is:

$$\exp\left(\lambda \sum_{i=0}^{i_0} a_i\, l^{\frac{i-q}{p}}\right)\left[\exp\left(\lambda \sum_{i\geqslant i_0+1} a_i\, l^{\frac{i-q}{p}}\right)-\mathbf{I}\right]. \tag{13}$$

One can ask the following questions:

— Find the factor and measure of this approximation.

— Give an appropriate form for the approximate solution.

— If the approximate solution is from $\mathcal{L}$, find an approximation of it suitable for a computer.

In 2.3. we saw that the function of E. M. Wright have a special role. This is the reason that we shall cite some of the well known results (see [2], [10], [12], [13]) for the Wright's functions we need it in the later:

1. Expansion in a Taylor series of the function Φ

$$\Phi(\beta, \rho; z)=\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)\,\Gamma(\rho n+\beta)}, \quad \rho>0 \text{ or } -1<\rho<0.$$

2. Bounds of the function Φ

Proposition A. *Let* $\lambda=e^{\alpha i}$, $0<\sigma<1$, $\beta<1$, $t\geqslant 0$ *and* $|\alpha|<\frac{\pi}{2}(1-\sigma)$, *then*

$$|t^{\beta-1}\Phi(\beta, -\sigma; -\lambda t^{-\sigma})|\leqslant \frac{C}{2\pi\sigma}\Gamma\left(\frac{1-\beta}{\sigma}\right)$$

where

$$C=\cos^{\frac{\beta-1}{\sigma}}\left(\alpha+\frac{\sigma\pi}{2}\right)+\cos^{\frac{\beta-1}{\sigma}}\left(\alpha-\frac{\sigma\pi}{2}\right)$$

3. *Approximation of the function* $t^{-1}\Phi(0, -\sigma; -\lambda t^{-\sigma})$, $0<\sigma<1$, *in a neighbourhood of zero*

$$t^{-1}\Phi(0, -\sigma; -\lambda t^{-\sigma})<\left(\frac{\sigma\lambda}{t}\right)^{\frac{1}{1-\sigma}} \exp\left[-(1-\sigma)\,\sigma^{\frac{\sigma}{1-\sigma}}\left(\frac{\lambda}{t^\sigma}\right)^{\frac{1}{1-\sigma}}\right], \quad \frac{\lambda}{t^\sigma}>\frac{2}{\sigma^\sigma}, \quad \lambda>0$$

$$t^{-1}\Phi(0, -\sigma; -\lambda t^{-\sigma})\leqslant \frac{T^{2n}}{(2n)!}C(2n+1)+\frac{T^{2n+1}}{(2n+1)!}C(2n+2)$$

for every $n\in\mathcal{N}$, $0\leqslant t\leqslant T$, where

$$C(k)=\frac{1}{\sigma\pi}\Gamma\left(\frac{k}{\sigma}\right)\left[\cos^{\frac{k}{\sigma}}\left(\alpha-\frac{\sigma\pi}{2}\right)+\cos^{\frac{k}{\sigma}}\left(\alpha+\frac{\sigma\pi}{2}\right)\right]$$

We need also a bound for $|a_i|$, $i \geqslant i_0+1$, where a_i are coefficients in (13):

This bound can be found in different manner for different special cases. So it can be used the Cauchy integral form and the inequality for the Taylor coefficients. We have only to introduce two new variables $v = wz^{q/p}$ and $u = z^{1/p}$ in (6). But in this case we need a bound M for the function defined by the Taylor series. In general case when we do not know it, we can use the following results: [9].

Let $D(z)$ be the polynomial in z obtained after the elimination of w from (7) and $\frac{\partial F(z, w)}{\partial w} = 0$. We denote by S_1, S_2 and S_3 the following sets:

$$S_1 \equiv \left\{z,\ \sum_{\nu=0}^{d_n} \alpha_{n,\nu} z^{m-\nu} = 0\right\},\quad S_2 \equiv \{z,\ D(z) = 0\},$$

$$S_3 \equiv \left\{z,\ \sum_{\nu=0}^{d_0} \alpha_{0,\nu} z^{m-\nu} = 0\right\}.$$

Proposition B. *Let (K, g) be a cannonical element which satisfies equation (7) with its center in the point $a \notin S_3$. We suppose that the radius ρ of the circle K is: $\rho < \min|z-a|$, $z \in \bigcup_{i=1}^{3} S_i$ then for the Taylor series coefficients a_n of the cannonical element (K, g) we have:*

$$|a_n| \leqslant \frac{4}{\rho} ne^2 |a_0| \left(1 - \frac{2}{n+k+1}\right)^{k-1} (n+k)^{2m-1},\ k \geqslant 0,\ n+k \geqslant 4$$

4.1. *Factor of the approximation*

The factor of the approximation in our case, as we can see from (13) is the product of elements analysed in 2.3.

If such an element belongs to $\mathcal{C}$ or $\mathcal{L}$ it is always expressed by one of the function of E. M. Wright and the cited properties can be used to make a valuation or an approximation of it by polynomials, which is very suitable for a computer.

4.2. *Approximation of the element* $\sum_{i \geqslant i_0+1} a_i l^{\frac{i-q}{p}}$

Our supposition is $i_0 > q$. To make a valuation of $\sum_{i \geqslant i_0+1} a_i l^{\frac{i-q}{p}}$ we shall use the found bound for $|a_i|$. In all cases there exist a M and a r such that $|a_i| \leqslant Mr^i$, $i \geqslant i_0$. Now

$$\left|\sum_{i \geqslant i_0+1} a_i l^{(i-q)/p} \leqslant \sum_{i \geqslant i_0+1} |a_i| l^{(i-q)p}\right.$$

$$\leqslant \sum_{j \geqslant 0} |a_{j+i_0+1}| l^{(j+i_0+1-q)/p}$$

$$\leqslant M r^{i_0+1} l^{(i_0+1-q)/p} \sum_{j\geqslant 0} (r l^{1/p})^j$$

$$\leqslant M r^{i_0+1} l^{(i_0+1-q)/p} \frac{\mathbf{I}}{\mathbf{I}-r l^{1/p}}$$

$$\leqslant M r^{i_0+1} l^{(i_0+1-q-p)/p} \left\{ p \int_0^{\infty} \Phi\left(0, -\frac{1}{p};\ -x t^{\frac{-1}{p}}\right) e^{rx} \frac{dx}{x} \right\}$$

For the last inequality see [3] p. 200, relation (4).

We can take an other estimation of our sum:

$$\sum_{i\geqslant i_0+1} a_i\, l^{(i-q)/p} \leqslant \sum_{j\geqslant 0} |a_{j+i_0+1}|\, l^{(j+i_0+1-q)/p}$$

$$\leqslant M l^{(i_0+1-q-p)/p} r^{i_0+1} \sum_{j\geqslant 0} r^j l^{(j+p)/p}$$

$$\leqslant_{\mathrm{T}} M l^{(i_0+1-q)/p} r^{i_0+1} \sum_{j\geqslant 0} r^j \frac{T^{j/p}}{\Gamma\left(\frac{j}{p}+1\right)},$$

where $\delta = \dfrac{i_0+1-q}{p}$ and $\nu \geqslant M r^{i_0+1} \sum_{j=0} r^j \dfrac{T^{j/p}}{\Gamma\left(\frac{j}{p}+1\right)}$.

It is easy to find a bound for ν.

4.3. *Measure of the approximation*

Now it is easy to find the measure of the approximation using the relation (13):

$$|\exp(\lambda \sum_{i\geqslant i_0+1} a_i\, l^{(i-q)/p}) - \mathbf{I}|$$

$$= |\sum_{k=1}^{\infty} \frac{1}{k!} (\lambda \sum_{i\geqslant i_0+1} a_i\, l^{(i-q)/p})^k|$$

$$\leqslant \sum_{k=1}^{\infty} \frac{1}{k!} |\lambda \sum_{i\geqslant i_0+1} a_i\, l^{(i-q)/p}|^k$$

$$\leqslant_{\mathrm{T}} \sum_{k=1}^{\infty} \frac{1}{k!} (|\lambda|\, \nu\, l^{\delta})^k \leqslant_{\mathrm{T}} \exp(|\lambda|\, \nu\, l^{\delta}) - \mathbf{I} = \{t^{-1}\, \Phi(0, \delta;\ |\lambda|\, \nu\, t^{\delta})\}$$

where ν and δ are given in 4.2.

For the approximation of the function $t^{-1}\Phi(0, \delta;\ |\lambda|\, \nu\, t^{\delta})$ by the Taylor polynomial see 4. relation 1. The bound of the factor of approximation, when it is a function, is given in 4. proposition **A**.

5. Application to partial differential equations

5.1. *Operator differential equation which corresponds to a partial differential equation*

Using the introduced notations, to the partial differential equation for the numerical functions

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu}\, x_{\lambda^{\mu},t^{\nu}}(\lambda, t) = \varphi(\lambda, t) \tag{14}$$

$\lambda_1 \leqslant \lambda \leqslant \lambda_2$, $0 \leqslant t < \infty$, corresponds operator differential equation:

$$\sum_{k=0}^{m} a_k(\mathbf{s})\, \mathbf{x}_{\lambda}{}^{k}(\lambda) = \mathbf{f}(\lambda), \ \lambda_1 \leqslant \lambda \leqslant \lambda_2, \tag{15}$$

where

$$a_k(\mathbf{s}) = \sum_{\nu=0}^{d_k} \alpha_{k,\nu}\, \mathbf{s}^{\nu}, \ d_k \leqslant n \qquad k = 0, 1, \ldots, m \tag{16}$$

and

$$\mathbf{f}(\lambda) = \{\varphi(\lambda, t\} + \sum_{k=0}^{n-1} \mathbf{s}^{n-k-1} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,n-k-\nu}\, x_{\lambda^{\mu},t^{\nu}}(\lambda, 0) \tag{17}$$

The exposed theory gives now the approximation of the linearly independent solutions of the homogeneous part of the equation (15), which we use to construct the solutions of the equation (15).

At the end let us remark:

1. The exposed theory can be applied only for partial differential equation of the form (14) with a domain: $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, $t \geqslant 0$.

2. Initial conditions are given in the sum (17). We do not need to know every addend $x_{\lambda^{\mu},t^{\nu}}(\lambda, 0)$ but all the sum

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,n-k-\nu}\, x_{\lambda^{\mu},t^{\nu}}(\lambda, 0), \qquad k = 0, 1, \ldots, n-1. \tag{18}$$

So, Cauchy initial condition is not equivalent to the given sums (18).

3. The solutions of the equation (15) are not always the solutions of the equation (14). Even then the solutions of the equations (15) are defined by functions; They can be solutions of the equation (14) in a generalized sense.

5.2. *Diffusions equation*

The equation

$$\frac{\partial U(\lambda, t)}{\partial t} = A \frac{\partial^2 U(\lambda, t)}{\partial \lambda^2} + B \frac{\partial U(\lambda, t)}{\partial \lambda}, \ A > 0 \tag{19}$$

is the first diffusions equation. We shall apply our theory to it. To the equation (19) with the initial condition $U(0, \lambda)=0$ corresponds in $\mathcal{M}$

$$A\frac{d^2\mathbf{u}(\lambda)}{d\lambda^2}+B\frac{d\mathbf{u}(\lambda)}{d\lambda}-\mathbf{s}\mathbf{u}(\lambda)=0 \tag{20}$$

The relation (6) in this case is:

$$A l\mathbf{w}^2+Bl\mathbf{w}-1=0 \text{ or } A\omega^2+B\mathbf{u}\omega-1=0\ (\omega=l^{1/2}\mathbf{w},\ l^{1/2}=\mathbf{u}).$$

Whence

$$\omega=\frac{-B\mathbf{u}\pm\sqrt{B^2\mathbf{u}^2+4A}}{2A}$$

$$=-\frac{B}{2A}\pm\frac{1}{\sqrt{A}}\sum_{k=0}^{\infty}\binom{1/2}{k}\left(\frac{B\mathbf{u}}{2\sqrt{A}}\right)^{2k}$$

and

$$\mathbf{w}=\pm\frac{1}{\sqrt{A}}l^{-1/2}-\frac{B}{2A}\pm\frac{1}{\sqrt{A}}\sum_{k=1}^{\infty}\binom{1/2}{k}\left(\frac{B^2}{4A}\right)^{k}l^{k-1/2}.$$

The linearly independent solutions of the equation (20) are given by:

$$\mathbf{u}_{1,2}(\lambda)=\exp\left[\pm\lambda\left(\frac{1}{\sqrt{A}}l^{-1/2}\mp\frac{B}{2A}+\frac{1}{\sqrt{A}}\sum_{k=1}^{\infty}\binom{1/2}{k}\left(\frac{B^2}{4A}\right)^{k}l^{k-1/2}\right)\right] \tag{21}$$

The general solution of the equation (20) is:

$$\mathbf{u}(\lambda)=\mathbf{c}_1\mathbf{u}_1(\lambda)+\mathbf{c}_2\mathbf{u}_2(\lambda)$$

$\mathbf{c}_1$ and $\mathbf{c}_2$ are elements from $\mathcal{M}$ which have to be determined by some additional conditions.

One approximation of operators (21) is

$$\mathbf{u}_{1,2}(\lambda)\approx e^{-\lambda B/2A}\exp\left[\pm\frac{\lambda}{\sqrt{A}}\left(l^{-1/2}+\sum_{k=1}^{i_0}\binom{1/2}{k}\left(\frac{B^2}{4A}\right)^{k}l^{k-1/2}\right)\right]$$

If λ is such that $\frac{\lambda}{\sqrt{A}}=-\mu$ or $-\frac{\lambda}{\sqrt{A}}=-\mu$, $\mu>0$ then the correspondent $\mathbf{u}_k(\lambda)$ is a function and

$$\mathbf{u}_k(\lambda)\approx e^{-\lambda B/2A}\left\{t^{-1}\Phi\left(0,\ -\frac{1}{2};\ -\mu t^{-1/2}\right)\right\}\times$$

$$\times\prod_{k=1}^{i_0}\left[\left\{t^{-1}\Phi\left(0,k-\frac{1}{2},\ -\mu\binom{1/2}{k}\left(\frac{B^2}{4A}\right)^{k}t^{k-1/2}\right)\right\}+\mathbf{I}\right]$$

The coefficients $\binom{1/2}{k}\left(\frac{B^2}{4A}\right)^{k}$ are with the alternate sign so:

$$\sum_{k=i_0+1}^{\infty}\binom{1/2}{k}\left(\frac{B^2}{4A}\right)^{k}l^{k-1/2}\leqslant_{\mathrm{T}}\left|\binom{1/2}{i_0+1}\right|\left(\frac{B^2}{4A}\right)^{i_0+1}l^{i_0+\frac{1}{2}},\ i_0\geqslant 1$$

The numbers δ and ν from 4.2. are $\delta = i_0 + 1/2$, $\nu = \binom{1/2}{i_0+1}\left(\frac{B^2}{4A}\right)^{i_0+1}$. Now it is easy to find the measure of the approximation.

We remind that the product in the expression for $\mathbf{u}_k(\lambda)$ is the second operation in the field $\mathcal{M}$ and that its restriction on $\mathcal{L}$ is the finite convolution.

REFERENCES

[1] Appel, P., Goursat, E., *Théorie des fonctions algébriques*, Paris (1929).

[2] Gajić, Lj., Stanković, B., *Neke osobine funkcije E. M. Wright-a*, Publications Inst. Math. Beograd 20 (1976), 91—98.

[3] Despotović, D., Stanković, B., *Continuité d'une fonction opératoire*, Publications Inst. Math., Beograd, 7 (1967), 197—203.

[4] Mikusiński, J., *Operational Calculus*, Warszawa (1959).

[5] Mikusiński, J., *Sur les fonctions exponentielles du calcul opératoire*, Studia Math. XII (1951), 208—224.

[6] Mikusiński, J., *Sur les notions de distributions et d'opérateur*, Bull. Ac. Polonaise des Sciences VI № 12 (1958), 737—741.

[7] Magnus, W., Oberhettinger, F., *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*, Berlin (1948).

[8] Stanković, B., *O jednoj klasi operatora*, Zbornik radova Prirodnomatematičkog fakulteta Novi Sad, v. 5 (1975), 9—19.

[9] Stanković, B., *Majoracija koeficijenata Tajlorovog reda kanoničnog elementa algebarske funkcije*, Zbornik radova Prirodnomatematičkog fakulteta, v. 6 (1976), 15—18.

[10] Stanković, B., *On the function of E. M. Wright*, Publications Inst. Math., 10 (1970), 113—124.

[11] Takači, A., *O jednoj klasi operatora*, VI Kongres matematičara, fizičara i astronoma Jugoslavije (1975)

[12] Wright, E. M., *The generalized Bessel function of order greater than one*, Quart J. Math. Oxford series, 2 (1940), 36—48).

]13] Wright, E. M., *On the coefficient of power series having exponential singularities*, Journal Lond. Math. Soc. 8 (1933), 71—79.