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TRANSITIVITY, ASSOCIATIVITY AND BISYMMETRY EQUATIONS ON *GD*-GROUPOIDS

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It is known [1], [2], [3] that if G is a non-empty set and the quasigroup operations defined on G satisfy the generalized equation of associativity or transitivity, for elements in G, then the quasigroups are isotopic to a group. Also, if the generalized bisymmetry equation holds, the quasigroups are isotopic to an Abelian group.

These results have been extended to the case of generalized associativity and bisymmetry equations on GD-groupoids. In this paper we consider the generalized transitivity equation on GD-groupoids and obtain results similar in nature to those found in [1], [2], [3] and also obtain results concerning the generalized associativity and bisymmetry equations on GD-groupoids in [4], [5] by reducing them to the transitivity equation.

Definitions. Consider non-empty sets S_1, S_2, S and a mapping $T: S_1 \times S_2 \to S$. Then the ordered quadruple $(S_1, S_2, S; T)$ is called a generalized groupoid or a G-groupoid. Also, for the sake of brevity, we occasionally become less formal and merely refer to the mapping T as a G-groupoid or as a G-groupoid on S_1, S_2 and S. If for every $a \in S_1$, $b \in S_2$, $c \in S$, the equations $T(x, b) = c$, $T(a, y) = c$ are solvable for $x \in S_1$ and $y \in S_2$, then the G-groupoid T is called a GD-groupoid. In the case when the solutions are unique, the GD-groupoid T is called a G-quasigroup. We use the following notations.

$$R_T(b)\, x = T(x, b), \qquad L_T(a)\, y = T(a, y).$$

A G-groupoid $(S_1, S_2, S; T)$ is homotopic to a G-groupoid $(S_1', S_2', S'; T')$ if there exist three surjections $\alpha: S_1 \to S_1'$, $\beta: S_2 \to S_2'$, $\nu: S \to S'$ such that

$$\nu\, T(x, y) = T'(\alpha x, \beta y) \quad \text{for every} \quad x \in S_1,\ y \in S_2,$$

(in which case the ordered triple $[\alpha, \beta, \nu]$ is called a homotopy).

The generalized equation of transitivity

Theorem 1. *Let* $(S_1, S_3, S_4; B)$, $(S_2, S_3, S_5; C)$, $(S_4, S_5, S; A)$ *and* $(S_1, S_2, S; D)$ *be GD-groupoids satisfying the generalized transitivity equation*

$$A(B(x, u), C(y, u)) = D(x, y) \tag{1}$$

for all $x \in S_1$, $y \in S_2$, $u \in S_3$ *with the mappings* $L_D(x): S_2 \to S$, $R_A^{\cdot}(v) = S_4 \to S$ *and* $R_C(u): S_2 \to S_5$ *as bijections for each* $x \in S_1$, $v \in S_5$, $u \in S_3$. *Then there is a group* $S(\bar{\circ})$ *homotopic to each of the given GD-groupoids.*

Proof. Put $u = e \in S_3$ in (1) to get

(2) $$A(R_b(e)x, R_C(e)y) = D(x, y), \text{ for } x \in S_1, y \in S_2.$$

Choose $t \in S_5$, $a \in S_1$ arbitrarily and fix them. Then define an operation ($\circ$) on S as follows.

(3) $$s \bar{\circ} r = A(R_A(t)^{-1}s, R_C(e)L_D(a)^{-1}r), \text{ for } s, r \in S.$$

This can be rewritten as

(4) $$A(p, q) = R_A(t)p \circ L_D(a)R_C(e)^{-1}q, \text{ for } p \in S_4, q \in S_5.$$

Since $R_A(t)$, $L_D(a)$, $R_C(e)$ are bijections, the operation ($\circ$) is well-defined on the set S. Also, $S(\circ)$ being the homotopic image of the *GD*-groupoid A under the homotopy $H = [R_A^{\cdot}(t), L_D(a)R_C(e)^{-1}, 1]$, $S(\circ)$ is itself a *GD*-groupoid [4]. Now, we will prove that $S(\circ)$ is a quasigroup. It is enough to show the uniqueness of the solution of equations, that is, the left and the right cancellativity of ($\circ$). *Let* $s_1 \circ r = s_2\ r$. Then, since $R_A(v)$ is a bijection, by (3) it follows that $R_A(t)^{-1}s_1 = R_A(t)^{-1}s_2$, which in turn yields $s_1 = s_2$, proving the right-cancellativity of ($\dot{\circ}$).

Now let $s \circ r_1 = s \circ r_2$. Then by (3) we get

(5) $$A(R_A^{\cdot}(t)^{-1}s, R_C(e)L_D(a)^{-1}r_1) = A(R_A^{\cdot}(t)^{-1}s, R_C(e)L_D(a)^{-1}r_2).$$

But since B is a *GD*-groupoid, there exists $x \in S_1$ such that $B(x, e) = R_A^{\cdot}(t)^{-1}s$, so that (5) becomes

(6) $$A(B(x, e), C(L_D(a)^{-1}r_1, e)) = A(B(x, e), C(L_D(a)^{-1}r_2, e)).$$

Now (6) and (1) yield

(7) $$D(x, L_D(a)^{-1}r_1) = D(x, L_D(a)^{-1}r_2).$$

Since $L_D(x)$ is a bijection, it follows from (7) that $L_D(a)^{-1}r_1 = L_D(a)^{-1}r_2$, that is $r_1 = r_2$, showing that ($\circ$) is left cancellative. Thus $S(\circ)$ is a quasigroup.

From (2) by means of (4), we get

(8) $$D(x, y) = R_A(t)R_B(e)x \bar{\circ} L_D(a)y.$$

By hypothesis, for every $u \in S_3$, $v \in S_5$, there is a unique $y \in S_2$ such that

(9) $$C(y, u) = v,$$

or there exists $H: S_3 \times S_4 \to S_2$ such that

(10) $$H(u, v) = y.$$

Substituting (9) and (10) into (1), we get

(11) $$A(B(x, u), v)=D(x, H(u, v)).$$

Setting $v=t\in S_5$ in (11), we have

(12) $$R_A(t)\,B(x, u)=D(x, R_H(t)\,u).$$

By putting $x=a\in S_1$ and $v=t\in S_5$ in (11), we get

(13) $$R_A(t)\,L_B(a)=L_D(a)\,R_H(t).$$

From (12), using (8) and (13) we get

(14) $$B(x, u)=R_A(t)^{-1}(R_A(t)\,R_B(e)\,x\circ R_A(t)\,L_B(a)\,u).$$

For $x=a$, (11) gives

$$A(L_B(a)\,u, v)=L_D(a)\,H(u, v),$$

which by the use of (4) yields

(15) $$H(u, v)=L_D(a)^{-1}(R_A(t)\,L_B(a)\,u\circ L_D(a)\,R_C(e)^{-1}\,v).$$

Substituting (4), (8), (14) and (15) into (11), we obtain

$$(R_A(t)\,R_B(e)\,x\circ R_A(t)\,L_B(a)\,u)\circ(L_D(a)\,R_C(e)^{-1}\,v)$$
$$R_A(t)\,R_B(e)\,x\circ(R_A(t)\,L_B(a)\,u\circ L_D(a)\,R_C(e)^{-1}\,v).$$

This shows that operation $(\circ)$ is associative. Thus $S(\circ)$ is a group. From (10) and (15), using $S(\circ)$ as a group, we have

$$y=L_D(a)^{-1}(R_A(t)\,L_B(a)\,u\circ L_D(a)\,R_C(e)^{-1}\,v),$$

that is

$$L_D(a)\,y=R_A(t)\,L_B(a)\,u\circ L_D(a)\,R_C(e)^{-1}\,v,$$

so that, (9) gives

(16) $$C(y, u)=v=R_C(e)\,L_D(a)^{-1}((R_A(t)\,L_B(a)\,u)^{-1}\circ L_D(a)\,y).$$

This could be rewritten as

(17) $$(L_D(a)\,y)^{-1}\circ(R_A(t)\,L_B(a)\,u)=(L_D(a)\,R_C(e)^{-1}\,C(y, u))^{-1}.$$

From (4), (8), (14) and (17), it follows that A, D, B and C are homotopic to the group $S(\circ)$. This completes the proof of the theorem.

With $R_A(t)=\alpha$, $R_A(t)\,R_B(e)=\lambda$, $L_D(a)\,R_C(e)^{-1}=\beta$,

$L_D(a)=\delta$ and $R_A(t)\,L_B(a)=\varphi$, (4), (8) (14) and (16) can be written as

$$\begin{cases} A(p, q)=\alpha p\circ\beta q. \\ D(x, y)=\lambda x\circ\delta y. \\ B(x, u)=\alpha^{-1}(\lambda x\circ\varphi u). \\ C(y, u)=\beta^{-1}((\varphi u)^{-1}\circ\delta y). \end{cases}$$

We introduce an equivalence relation in the set of all surjections from a set M onto a set N in the following way. $\alpha\sim\beta$ if there exist fixed elements $a, b\in N$ such that for every $x\in M$, $\alpha x=a*\beta x*b$, where $N(*)$ is a group. Thus we have the following theorem.

Theorem 2. *If the four groupoids A, B, C and D satisfy the hypotheses of theorem 1, then the general solution of equation (1) is*

$$(18)\qquad \begin{cases} A(x, y)=\alpha x\circ\beta y \\ D(x, y)=\lambda x\circ\delta y \\ B(x, y)=\alpha^{-1}(\lambda x\circ\varphi y) \\ C(x, y)=\beta^{-1}((\varphi y)^{-1}\circ\delta x). \end{cases}$$

where ($\circ$) is a group which is unique upto isomorphism, and the mapping α, β, λ, δ, φ are unique upto equivalence. Conversely, if A, B, C, D are of the form (18) *then they satisfy equation* (1).

The generalized associativity equation.

We now use Theorem 1 to obtain the following known result (see [4]).

Theorem. 3. *Let $(S_4, S_5, S; A)$; $(S_1, S_3, S_4; B)$, $(S_3, S_5, S_2; H)$, and $(S_1, S_2, S; D)$ be GD-groupoids. If the generalized associativity equation*

$$(19)\qquad A(B(x, u), z)=D(x, H(u, z))$$

holds for all $x\in S_1$, $u\in S_3$, $z\in S_5$ and if the mappings $L_D(x): S_2\to S$, $L_H(u): S_5\to S_2$, $R_A(v): S_4\to S$ are bijections for each $x\in S_1$, $u\in S_3$, $v\in S_5$, then there is a group $S(\circ)$ homotopic to each of the above GD-groupoids.

Remark: In Theorem 3, we assume that $L_H(u)$ is a bijection for each $u\in S_3$, which is not required in [4]. It is not possible without this additional condition to pass from the associativity equation to the transitivity equation which is the method adopted in the following proof.

Proof. From the above hypothesis it follows that for each $u\in S_3$ and each $y\in S_2$ there is a unique $z=S_5$ so that

$$(20)\qquad H(u, z)=y$$

and so there exists a mapping $C: S_2\times S_3\to S_5$ such that

$$(21)\qquad C(y, u)=z.$$

Also it is easy to see that $(S_2, S_3, S_5; C)$ is a GD-groupoid. Then (19) reduces to

$$(22)\qquad A(B(x, u), C(y, u))=D(x, y)$$

for all $x\in S_1$, $u\in S_3$, $y\in S_2$ which is merely the generalized transitivity equation (1). From (20) and (21) it follows that $R_C(u)=L_H(u)^{-1}$ for each $u\in S_3$. So, in addition to $R_A(v)$ and $L_D(x)$ being bijections, it is also true that $R_C(u)$ is a bijection. Hence, we can appeal to Theorem 1 and conclude that there is a group $S(\circ)$ homotopic to each of the GD-groupoids A, B, C and D.

Now let $[\alpha, \beta, \gamma]$ be a homotopy of C onto $S(\circ)$. Then

$$(23)\qquad \alpha y\circ\beta u=\gamma C(y, u)$$

for all $y\in S_2$ and all $u\in S_3$. For each $u\in S_3$ and $z\in S_5$ there is a unique $y\in S_2$ so that $C(y, u)=z$ and $H(u, z)=y$. Then (23) becomes $\alpha H(u, z)\circ\beta u=\gamma z$ and so

$$\beta u\circ(\gamma z)^{-1}=(\alpha H(u, z))^{-1} \tag{24}$$

for all $u\in S_3$ and all $z\in S_5$. Hence, H is also homotopic to the group $S(\circ)$ and the proof of Theorem 3 is complete.

Further more, one can now use the general solution for (22) obtained in the preceding section in conjunction with (20) and (21) to get the following general solution to (19):

$$\begin{aligned} A(p, q)&=\alpha p\circ\beta q\\ D(x, y)&=\lambda x\circ\delta y\\ B(x, u)&=\alpha^{-1}(\lambda x\circ\varphi u)\\ H(u, z)&=\delta^{-1}(\varphi u\circ\beta z)\end{aligned} \tag{25}$$

where $\alpha=R_A(t)$, $\beta=L_D(a)L_H(e)$, $\lambda=R_A(t)R_B(e)$, $\delta=L_D(a)$, and $\varphi=R_A(t)L_B(a)$ for any fixed $e\in S_3$, $t\in S_5$ and $a\in S_1$.

Thus we have proved the following theorem.

Theorem 4. *If the four GD-groupoids A, B, H and D satisfy the conditions of Theorem 3, then the general solution of equation (19) is given by (25) where $S(\circ)$ is a group and α, β, λ, δ, φ are mappings from S_4, S_5, S_1, S_2 and S_3 into S, S_2, S_4, S and S_4 respectively. Conversely, if $S(\circ)$ is a group and A, B, H and D are defined as in (25) then equation (19) is satisfied.*

The generalized equation of bisymmetry.

We now employ Theorem 1 to obtain the following known result (see [5]).

Theorem 5. *Let $(S_5, S_6, S; A_1)$ and $(S_7, S_8, S; B_1)$ be G-quasigroups and let $(S_1, S_3, S_5; A_2)$, $(S_2, S_4, S_6; A_3)$, $(S_1, S_2, S_7; B_2)$ and $(S_3, S_4, S_8; B_3)$ be GD-groupoids. If the bisymmetry equation*

$$A_1(A_2(x, u), A_3(y, v))=B_1(B_2(x, y), B_3(u, v)) \tag{26}$$

holds for all $x\in S_1$, $u\in S_3$, $y\in S_2$, $v\in S_4$ and if the mappings $L_{B_3}(u): S_4\to S_8$, and $R_{A_3}(v): S_2\to S_6$ are bijections for each $u\in S_3$ and each $v\in S_4$, then there is an abelian group $S(\circ)$ which is homotopic to each of A_i and B_i for $i=1, 2, 3$.

Remark 2. In Theorem 5, it is assumed that $L_3(u)$ and $R_3(v)$ are bijections (which are not used in [5]) which are necessary in order to reduce equation (26) to the generalized transitivity equation (1).

Proof. From the hypothesis, for each $u\in S_3$ and each $w\in S_8$ there exists a unique $v\in S_4$ such

$$B_3(u, v)=w, \tag{27}$$

and so there exists a mapping $K: S_8 \times S_3 \to S_4$ such that

$$K(w, u) = v. \tag{28}$$

By (27) and (28), (26) reduces to

$$A_1(A_2(x, u), A_3(y, K(w, u))) = B_1(B_2(x, y), w).$$

In this equation setting $w = e \in S_8$, we get

$$A_1(A_2(x, u), A_3(y, L_k(e)u)) = R_{B_1}(e) B_2(x, y) \tag{29}$$

and this could be rewritten as:

$$A_1(A_2(x, u), A_3'(y, u)) = B_2'(x, y) \tag{30}$$

for all $x \in S_1$, $u \in S_3$ and $y \in S_2$, which is merely the generalized transitivity equation (1) where

$$A_3'(y, u) = A_3(y, L_K(e)u)$$

(31) and

$$B_2'(x, y) = R_{B_1}(e) B_2(x, y).$$

It is easy to see that $(S_2, S_3, S_6; A_3')$ and $(S_1, S_2, S; B_2')$ are *GD*-groupoids. Fixing in (26), one after the other, $x=a$, $u=c$, $y=b$; $x=a$, $y=b$, $v=d$; $u=c$, $y=b$, $v=d$; $x=a$, $u=c$, $v=d$ and for brevity denoting the left and right translations of the *G*-quasigroup A_1 (or B_1) by simply L_{A_1} (or L_{B_1}) and R_{A_1} (or R_{B_1}) respectively, we get

$$\begin{cases} L_{A_1} L_{A_3}(b) = L_{B_1} L_{B_3}(c), & R_{A_1} L_{A_2}(a) = L_{B_1} R_{B_3}(d), \\ R_{A_1} R_{A_2}(c) = R_{B_1} R_{B_2}(b), & L_{A_1} R_{A_3}(d) = R_{B_1} L_{B_2}(a). \end{cases} \tag{32}$$

Since A_1 and B_1 are *G*-quasigroups, by hypothesis, from (32) we get

$$L_{B_2}(a) = R_{B_1}^{-1} L_{A_1} R_{A_3}(d) \text{ is a bijection for all } a \in S_1. \tag{33}$$

Further, (30) when $x=a$ and $u=c$ yields

$$L_{A_1} R_{A'_3}(c) = L_{B'_2}(a). \tag{34}$$

When $x=a$, (31) gives

$$L_{B'_2}(a) = R_{B_1} L_{B_2}(a). \tag{35}$$

This, with (33) implies $L_{B'_2}(x)$ is a bijection for all $x \in S_1$. Moreover, from (31), when $u = c \in S_3$, we get

$$R_{A'_3}(c) = R_{A_3}(L_k(e)(c)). \tag{36}$$

Since $R_{A_3}(v): S_2 \to S_6$ is a bijection for all $v \in S_4$, from (36) $R_{A'_3}(u)$ is a bijection for all $u \in S_3$.

Thus, by applying Theorem 1 to the equation (30), each of A_1, A_2, A'_3 and B'_2 are homotopic to a group $S(\circ)$ and, in view of the information given in connection with Theorem 1 as well as (31), (32), (34) and (36), we obtain the following expressions.

(37) $$A_1(p, q) = R_{A_1}p \circ L_{A_1}q, \text{ for } p \in S_4,\ q \in S_6$$

(38) $$B'_2(x, y) = R_{A_1}R_{A_2}(c)\,x \circ L_{B'_2}(a)\,y = R_{A_1}R_{A_2}(c)\,x \circ L_{A_1}R_{A'_3}(c)\,y, \text{ for } x \in S_1,\ y \in S_2$$

(39) $$A_2(x, u) = R_{A_1}^{-1}(R_{A_1}R_{A_2}(c)\,x \circ R_{A_1}L_{A_2}(a)\,u) = R_{A_1}^{-1}(R_{B_1}R_{B_2}(a)\,x \circ L_{B_1}R_{B_3}(d)\,u), \text{for } x \in S_1, u \in S_3.$$

(40) $$A'_3(y, u) = R_{A'_3}(c)\,L_{B'_2}(a)^{-1}((R_{A_1}L_{A_2}(a)\,u)^{-1} \circ L_{B'_2}(a)\,y), \text{ for } u \in S_3,\ y \in S_2.$$

Now (38) in conjunction with (34), (35) and (32) gives,

(41) $$B_2(x, y) = R_{B_1}^{-1}(R_{A_1}R_{A_2}(c)\,x \circ L_{A_1}R_{A_3}(d)\,y), \text{ for } x \in S_1,\ y \in S_2,$$

showing thereby that B_2 is homotopic to $S(\circ)$.

Put $x = a$ and $y = b$ in (26). Then by (37) we have

(42) $$B_3(u, v) = L_{B_1}^{-1}(R_{A_1}L_{A_2}(a)\,u \circ L_{A_1}L_{A_3}(b)\,v), \text{ for } u \in S_3,\ v \in S_4.$$

Thus B_3 is also homotopic to $S(\circ)$.

When $u = c$ and $y = b$ equation (26) with (32) yields

$$A_1(R_{A_2}(c)\,x, L_{A_3}(b)\,v) = A_1(R_{A_1}^{-1}R_{B_1}R_{B_2}(b)\,x, L_{A_1}^{-1}L_{B_1}L_{B_3}(c)\,v).$$

By setting $R_{B_2}(b)\,x = s$, $L_{B_3}(c)\,v = t$, we get

(43) $$B_1(s, t) = A_1\left(R_{A_1}^{-1}R_{B_1}s, L_{A_1}^{-1}L_{B_1}t\right) = R_{B_1}s \circ L_{B_1}t.$$

Hence B_1 is also homotopic to $S(\circ)$.

Substituting (37), (39), (40), (41), (42) and (43) into (26) we get

(44) $$(R_{B_1}R_{B_2}(b)\,x \circ L_{B_1}R_{B_3}(d)\,u) \circ ((R_{A_1}L_{A_2}(a)\,u)^{-1} \circ L_{B'_2}(a)\,y) = (R_{A_1}R_{A_2}(c)\,x \circ L_{A_1}R_{A_3}(d)\,y) \circ (R_{A_1}L_{A_2}(a)\,u \circ L_{A_1}L_{A_3}(b)\,v).$$

With $R_{B_1}R_{B_2}(b)\,x = \xi$, $L_{B_1}R_{B_3}(d)\,u = \eta$, $(R_{A_1}L_{A_2}(a)\,u)^{-1} = \theta$, $L_{A_1}L_{A_3}(b)\,v = \theta'$, $L_{B_2'}(a)\,y = \delta$, (44) becomes

$$(\xi \circ \eta) \circ (\theta \circ \delta) = (\xi \circ \delta) \circ (\eta \circ \theta').$$

Taking $\xi = \eta = \delta =$ identity, we get first $\theta = \theta'$ which leads to

$$\eta \circ \delta = \delta \circ \eta.$$

Hence $(\circ)$ is commutative. Therefore $S(\circ)$ is an abelian group.

$$(R_{A_1}L_{A_2}(a)\,u)^{-1} = L_{A_1}L_{A_3}(b)\,v.$$

Substituting into (40), using (32) we get

$$(45)\qquad \begin{aligned} A_3(y, v) &= R_{A'_3}(c)\, L_{B'_2}(a)^{-1}\,(L_{A_1} L_{A_3}(b)\, v \circ L_{B'_2}(a)\, y) \\ &= R_{A'_3}(c)\, L_{B'_2}^{-1}(a)^{-1}\,(L_{B'_2}(a)\, y \circ L_{A_1} L_{A_3}(b)\, v) \\ &= L_{A_1}^{-1}\,(R_{B_1} L_{B_2}(a)\, y \circ L_{B_1} L_{B_3}(c)\, v) \text{ for } y \in S_2,\ v \in S_4, \end{aligned}$$

so that A_3 is also homotopic to $S(\circ)$. This completes the proof of the Theorem. With $R_{A_1}=f$, $L_{A_1}=g$, $R_{B_1}=h$, $L_{B_1}=k$, $R_{B_1}R_{B_2}(b)=R_{A_1}R_{A_2}(c)=\varphi$,

$$L_{B_1}R_{B_3}(d)=R_{A_1}L_{A_2}(a)=\psi,\ L_{A_1}R_{A_3}(d)=R_{B_1}L_{B_2}(a)=\alpha,$$

$L_{B_1}L_{B_3}(c)=L_{A_1}L_{A_3}(b)=\beta$, from (37), (39), (41), (42), (43) and (45) we get

$$(46)\qquad \begin{cases} A_1(x, y) = fx \circ gy. \\ A_2(x, y) = f^{-1}(\varphi x \tilde{\circ} \psi y). \\ B_2(x, y) = h^{-1}(\varphi x \circ \alpha y). \\ B_3(x, y) = k^{-1}(\psi x \circ \beta y). \\ B_1(x, y) = hx \circ ky. \\ A_3(x, y) = g^{-1}(\alpha x \circ \beta y). \end{cases}$$

Thus the following Theorem is proved.

Theorem 6. *Let the GD-groupoids A_i, B_i $(i=1, 2, 3)$ defined as in Theorem 5 satisfy equation (26). Then, under the same hypotheses as in Theorem 5, the general solution of (26) is given by (46), where $S(\tilde{\circ})$ is an abelian group. Conversely, if $S(\circ)$ is an abelian group and A_i, B_i $(i=1, 2, 3)$ are defined as in (46) with eight maps $f, g, h, k, \varphi, \psi, \alpha$, and β given, then A_i, B_i $(i=1, 2, 3)$ satisfy the generalized bisymmetry equation (26).*

Now as an application of theorem 1, we consider the following functional equation

$$(47)\qquad A(B(x_1^{k-1}, x_k^{m-1}), C(x_n, x_k^{n-1})) = D(x_1^{k-1}, x_n),$$

where A, B, C, D are quasigroups of different arities defined on the same non-empty set S of arities $|A_2|=2$, $|B|=n-1$, $|C|=n-k+1$, and $|D|=k$.

Denoting $B(x_1^{k-1}, x_k^{m-1}) = \tilde{B}((x_1^{k-1}, (x_k^{n-1})) = \tilde{B}(x, z)$, where $x=(x_1^{k-1})$, $z=(x_k^{n-1})$,

$$C(x_n, x_k^{n-1}) = \tilde{C}(y, z), \text{ where } y = x_n,$$

$$D(x_1^{k-1}, x_n) = \tilde{D}(x, y),$$

(47) could be written as,

$$(48)\qquad A(\tilde{B}(x, z), \tilde{C}(y, z)) = \tilde{D}(x, y),$$

where A is a quasigroup, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are GD-groupoids given by,

$$\tilde{B}: S^{k-1} \times S^{n-k} \to S,\quad \tilde{D}: S^{k-1} \times S \to S,\quad \tilde{C}: S \times S^{n-k} \to S,\quad \tilde{A}: S \times S \to S.$$

Evidently R_A, $L_{\bar{D}}$ and $R_{\bar{C}}$ are bijections. That is, equation (48) satisfies the conditions of theorems (1) and (2) and hence

$$(49)\qquad \begin{cases} A(p, q) = \alpha p \circ \beta q \\ B(x_1^{k-1}, x_k^n) = \alpha^{-1}(\gamma x_1^{k-1} \circ \varphi x_k^{n-1}) \\ C(x_n, x_k^{n-1}) = \beta^{-1}(\varphi x_k^{n-1} \circ \delta x_n) \\ D(x_1^{k-1}, x_n) = \gamma x_1^{k-1} \circ \delta x_n, \end{cases}$$

where α, β, γ, δ and φ are arbitrary permutation on S.

Thus, we have proved the following theorem.

Theorem 7. *Let A, B, C and D be quasigroups of arities $|A| = 2$, $|B| = n-1$, $|C| = n-k+1$ and $|D| = k$ defined on the same set S, satisfying equation* (47). *Then all solutions of* (47) *are given by* (49) *where α, β, γ, δ and φ are arbitrary permutations on S and* ($\circ$) *is an arbitrary group operation defined on S.*

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