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ON THE EMBEDDING OF Ω-ALGEBRAS IN GROUPOIDS

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Summary. It has been proved that any Ω-algebra can be embedded both in a semigroup [1], [3], and in a so called entropic groupoid [2].

This paper gives a necessary and sufficient condition for embedding any Ω-algebra in some groupoid $\mathcal{G}=(G, *)$ satisfying[1)] the set of laws $\Sigma(*)$. The condition is:

There is a term $\xi(x, y; *)$ formed of two variables x, y and the operation symbol $*$ such that the operation $\circ$ defined by

$$(\mathcal{C}) \qquad xy\circ \overset{\text{def}}{=} \xi(x, y; *)$$

does not satisfy any algebraic law (except the law $x=x$), while the operation $*$ satisfies the laws in $\Sigma(*)$.

Some new examples of groupoids which satisfy the condition ($\mathcal{C}$) are presented:

1° Each Ω-algebra can be embedded in a commutative groupoid.

2° Each Ω-algebra can be embedded in a groupoid satisfying the law of the type

$$\Pi\, x_1 \cdots x_n = \Pi\, x_{p_1} \cdots x_{p_n},$$

where at both sides of the equality the arrangement of the operation symbols is the same, and the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$ has at least one fixed point. For example, the laws of this type are

$$xy*z*=zy*x*, \quad xy*u*vw**=wy*v*ux**.$$

In the second part of the paper it is shown that the condition ($\mathcal{C}$) can be extended so that the main theorems 1, 2 hold for the laws $\Sigma(\mathcal{O})$, i. e. for variety $V_{\mathcal{O}}(\Sigma)$, where $\mathcal{O}$ is a set of some operation symbols. For example, any Ω-algebra can be embedded in an entropic algebra, i. e. in an algebra satisfying:

$$(x_{11}x_{12}\cdots x_{1n}f\cdots x_{n1}x_{n2}\cdots x_{nn}f)f=(x_{11}x_{21}\cdots x_{n1}f\cdots x_{1n}x_{2n}\cdots x_{nn}f)f$$

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1) In other words, $\mathcal{G}$ belongs to the variety $V_*(\Sigma)$, [1].

1. The main result of the paper is the following

Theorem 1. *If Q is an arbitrary Ω-algebra and $\Sigma(*)$ a set of laws*[1] *satisfying the condition $(\mathcal{C})$, then there exists a groupoid $\mathcal{G}=(G, *)$ satisfying the laws $\Sigma(*)$, with the properties:*

(i) *Q is a subset of G;*

(ii) *If $\omega\in\Omega$ is some n-ary operation, then there exists an $\overline{\omega}\in G$ and a term $t_\omega(x_1, \ldots, x_n, \overline{\omega}; \circ)$ formed from variables $x_1, \ldots, x_n$, the constant symbol $\overline{\omega}$ and the operation symbol $\circ$ such that:*

$$\omega=\overline{\omega}, \quad \text{if} \quad \omega\in\Omega(\circ)$$

$$x_1 x_2 \cdots x_n \omega = t_\omega(x_1, \ldots, x_n, \overline{\omega}; \circ), \quad \text{if} \quad \omega\in\Omega(n), \quad n=1, 2, \ldots$$

where $xy\circ \overset{\text{def}}{=} \xi(x, y; *)$. The existence of $\xi(x, y; *)$ is guaranteed by $(\mathcal{C})$.

For the proof we need the following definitions:

I. Let G be a *minimal set* satisfying the conditions:

The set $X\overset{\text{def}}{=}Q\cup\Omega$ is a subset of G,

If $u, v\in G$, then $uv* \in G$.

Let $\mathcal{G}=(G, *)$ be a groupoid determined by the set G and by the operation defined in the natural way, i. e. $\mathcal{G}$ is the $*$-word algebra on X.

II. Let G_0 be a *minimal subset of G* such that:

X is a subset of G_0,

If $u, v\in G_0$, then $uv\circ \in G_0$, where $\circ$ is defined by the term $\xi(x, y; *)$ (of condition $(\mathcal{C})$):

Def $(\circ)$ $$xy\circ \overset{\text{def}}{=} \xi(x, y; *)$$

III. For each $\omega\in\Omega(n)$ $(n=0, 1, \ldots)$ we define an *operation* $\boldsymbol{\omega}$ on G:

Def $(\boldsymbol{\Omega})$ $$\boldsymbol{\omega}=\omega, \text{ if } n=0,$$

$$x_1 \ldots x_n \boldsymbol{\omega} = t_\omega(x_1, \ldots, x_n, \omega; \circ), \quad \text{if} \quad n=1, 2, \ldots,$$

where $t_\omega(x_1, \ldots, x_n, \omega; \circ)$ is a term formed from variables $x_1, \ldots, x_n$, the constant ω and the operation symbol $\circ$. These terms can be chosen arbitrarily, in particular, they can be chosen with the operation symbols grouped on the right [2]. The operations defined in that way form the set

$$\boldsymbol{\Omega}\overset{\text{def}}{=}\{\boldsymbol{\omega} \mid \omega\in\Omega\}.$$

IV. Finally, let G_Ω be a *minimal subset* of G_0 such that:

$Q\cup\Omega(0)$ is a subset of G_Ω,

If $u_1, \ldots, u_n\in G_\Omega$, and $\omega\in\Omega(n)$ $(n=1, 2, \ldots)$, then $u_1 \ldots u_n \boldsymbol{\omega}\in G_\Omega$. Hence, the elements of G_Ω are those terms of G which can be represented by operations from $\boldsymbol{\Omega}$.

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[1] We assume that the laws are consistent, i. e. that they do not imply $x=y$, where x, y are different variables.

V. Let $\sim_Z$, $\sim_Q$, $\sim_\circ$, $\sim_\Omega$ be the *minimal congruences* generated by

$$\Sigma(*),\quad \operatorname{Tab} Q(\Omega),\quad \operatorname{Def}(\circ),\quad \operatorname{Def}(\mathbf{\Omega}),$$

that is:

$$u \sim_Z v \overset{\text{def}}{\Leftrightarrow} \Sigma(*) \vdash u = v,$$

$$u \sim_Q v \overset{\text{def}}{\Leftrightarrow} \operatorname{Tab} Q(\mathbf{\Omega}) \vdash u = v, \qquad (u, v \in G)$$

$$u \sim_\circ v \overset{\text{def}}{\Leftrightarrow} \operatorname{Def}(\circ) \vdash u = v$$

$$u \sim_\Omega v \overset{\text{def}}{\Leftrightarrow} \operatorname{Def}(\mathbf{\Omega}) \vdash u = v.$$

The symbol $\vdash$ denotes the *logical deduction* [4], for example $\Sigma(*) \vdash u = v$ means that the formula $u = v$ can be derived from the laws $\Sigma(*)$ and the equality axioms. $\operatorname{Tab} Q(\Omega)$ is the so called *positive diagram* [4] of the algebra Q, i. e. the set of all equalities of the form

$$a_1 \cdots a_n \omega = a \qquad (a_1, \ldots a_n, a \in Q,\ \omega \in \Omega(n),\quad n = 1, 2, \ldots)$$

which hold in Q. $\operatorname{Tab} Q(\mathbf{\Omega})$ is the corresponding set of formulas obtained by exchanging each $\omega \in \Omega$ by $\boldsymbol{\omega} \in \mathbf{\Omega}$.

VI. The relation $\sim$ is the *minimal congruence* of the set G generated by

$$\sim_Z,\ \sim_Q,\ \sim_\circ,\ \sim_\Omega.$$

Lemma. *Let ρ be one of the relations $\sim_Z$, $\sim_Q$, $\sim_\circ$, $\sim_\Omega$ and let σ be one of the relations $\sim_Z$, $\sim_\circ$, $\sim_\Omega$. Then*

$$(u \in G_\Omega \wedge u \rho v) \Rightarrow (\exists v' \in G_\Omega)\, v \sigma v'.$$

Proof. We distinguish four cases: $u \sim_Z v$, $u \sim_Q v$, $u \sim_\circ v$, $u \sim_\Omega v$.

Ad1. In this case the term v' is just u, since by condition ($\mathcal{C}$), each term can be uniquely represented by $\circ$ (if such a representation exists), and hence uniquely represented by operations from $\mathbf{\Omega}$ (if the representation exists).

Ad2. Let $u = u(\cdots a_1 \cdots a_n \boldsymbol{\omega} \cdots)$, $a_1, \ldots a_n \in Q$. If the subterm of u: $a_1 \cdots a_n \boldsymbol{\omega}$ is replaced by a, under the condition that (in Q) $a_1 \cdots a_n \omega = a$, the resulting term is again in G_Ω. Similarly, in the case $u\ (\cdots a \cdots)$, i. e. a is a subterm of u, we conclude that $u(\cdots a_1 \cdots a_n \boldsymbol{\omega} \cdots) \in G_\Omega$, if $a_1 \cdots a_n \omega = a$ holds in Q. Since $\sim_Q$ can be defined in a finite number of such replacements, we have

$$u \in G_\Omega \wedge u \sim_Q v \Rightarrow v \in G_\Omega.$$

In cases 3 and 4, it immediately follows that v' is uniquely determined and equal to u. Hence:

(1) *In the cases $u \sim_Z v$, $u \sim_\circ v$, $u \sim_\Omega v$, if $u \in D_\Omega$, then the uniquely determined u' is just u.*

Proof of Theorem 1. We first prove:

(2) $$\bar{x} = \bar{y} \Rightarrow x = y \qquad (x, y \in Q),$$

where $\bar{x}, \bar{y}$ are the equivalence classes of x, y with respect to $\sim$.

Suppose that $\bar{x}=\bar{y}$, i. e. $x\sim y$. Then there exists a natural number k and elements

$$u_1, u_2, \ldots, u_k; \quad u_1=x, \quad u_k=y \qquad (u_i \in G)$$

such that for each $i=1, \ldots, k-1$:

$$u_i \sim_Z u_{i+1} \text{ or } u_i \sim_Q u_{i+1} \text{ or } u_i \sim_\circ u_{i+1} \text{ or } u_i \sim_\Omega u_{i+1}.$$

Let **Int** be an interpretation, i. e. a mapping such that

(i) If $x \in Q$, then $\mathbf{Int}(x) \overset{\text{def}}{=} x$,

(ii) If $\omega \in \Omega(0)$, then $\mathbf{Int}(\omega) \overset{\text{def}}{=} \omega$,

(iii) If $\omega \in \Omega(n)$ $(n=1, 2, \ldots)$ and $t_1, \ldots t_n \in D_\Omega$, then

$$\mathbf{Int}(t_1 \cdots t_n \omega) \overset{\text{def}}{=} \mathbf{Int}(t_1) \cdots \mathbf{Int}(t_n)\, \omega$$

(iv) If $t \in D$ and there exists a $t' \in D_\Omega$ such that

$$\Sigma(*), \quad \mathrm{Def}(\circ), \quad \mathrm{Def}(\Omega) \vdash t=t'$$

then $\mathbf{Int}(t) \overset{\text{def}}{=} \mathbf{Int}(t')$.

The mapping **Int** is well defined. The conditions (i), (ii), (iii) represent the usual definition of homomorphism. The soundness of part (iv) follows from (1).

The mapping **Int** carries the sequence $u_1, \ldots, u_k$ into the sequence

$$\mathbf{Int}(u_1), \ \mathbf{Int}(u_2), \ldots, \ \mathbf{Int}(u_k)$$

of elements of Ω-algebra Q:

From the definition of the mapping **Int** we have:

If one of the conditions $u \sim_Z v$, $u \sim_Q v$, $u \sim_\circ v$, $u \sim_\Omega v$ is satisfied, then the equality $\mathbf{Int}(u)=\mathbf{Int}(v)$ holds in Q.

Hence:

$$x=\mathbf{Int}(u_1)=\mathbf{Int}(u_2)= \cdots =\mathbf{Int}(u_k)=y$$

implying that the equality $x=y$ holds in Q, which proves (2).

In order to complete the proof of the theorem, we introduce the quotient groupoid $\bar{\mathcal{G}}=(\bar{\mathrm{G}}, \bar{*})$, where

$$\bar{\mathrm{G}} \overset{\text{def}}{=} \{\bar{x} \mid x \in G\}, \quad \bar{x}\bar{y}\bar{*} \overset{\text{def}}{=} \overline{xy*}.$$

The groupoid $\bar{\mathcal{G}}$ satisfies the same laws as $\mathcal{G}$.

Further, let

$$\bar{Q} \overset{\text{def}}{=} \{\bar{x} \mid x \in Q\}.$$

For each $\omega \in \Omega(n)$ we define an operation $\bar{\omega}$ in $\bar{Q}$ in the following way

$$\bar{\omega} \overset{\text{def}}{=} \bar{\omega}, \quad \text{if} \quad \omega \in \Omega(0)$$

$$\bar{x}_1 \cdots \bar{x}_n \bar{\omega} \overset{\text{def}}{=} \overline{x_1 \cdots x_n \omega}, \text{ if } \omega \in \Omega(n) \qquad (n=1, 2, \ldots).$$

It is clear that $\bar{Q}$ is an Ω-algebra ($\bar{\omega}$ corresponds to ω). From the first part of the proof it follows that Q and $\bar{Q}$ are isomorphic algebras (an isomorphism being $f: x \to \bar{x}$). This completes the proof of the theorem.

2. We give some examples of the laws $\Sigma(*)$ satisfying the condition $(\mathcal{C})$.

I. Already known examples of the laws $\Sigma(*)$ are associative and entropic laws ([1], [3], [2]). The simplest terms $\xi(x, y; *)$ in the first case are

$$ax * y *, \quad xy * a *$$

and in the second one

$$xa * ay **, \quad xx * xy **, \quad xy * yy **.$$

II. The commutative law also satisfies the condition $(\mathcal{C})$. One example of the term $\xi(x, y; *)$ is: $xx * xy **$. This is the well known term which is used to define the ordered pair.[1)]

III. If $\Sigma(*)$ is a law of the form

$$\Pi x_1 \cdots x_n = \Pi x_{p_1} \cdots x_{p_n}$$

where at both sides of the equality are terms with the same arrangement of operation symbols, and the permutation $p = \begin{pmatrix} 1 \cdots n \\ p_1 \cdots p_n \end{pmatrix}$ has at least one fixed point, for example $p(i) = i$, then one convenient term is

$$\Pi x \cdots xyx \cdots x \qquad (y \text{ is at the } i\text{-th place})$$

In the case of laws:

$$xy * z * = zy * x *, \qquad xy * u * vw ** = wy * v * ux **$$

such terms are: $xy * x *$, $xy * x * xx **$ respectively, for they stay unchanged after applying the corresponding laws [2].

IV. An example of laws of the previous type is

$$\Pi x_1 \cdots x_n = \Pi x_{p_1} \cdots x_{p_n}$$

under the condition that the permutation p has at least two fixed points, say $p(i) = i$, $p(j) = j$. The convenient term $\xi(x, y; *)$ can be formed as in III, as well as in different way. Namely

$$\Pi a \cdots axa \cdots aya \cdots a \qquad (a \text{ is a constant; } x \text{ and } y \text{ are at the } i\text{-th and } j\text{-th place})$$

satisfies also the condition $(\mathcal{C})$.

Remark. If the term $\xi(x, y; *)$ is such that it depends on the constant a, i. e. the term of the form $\xi(x, y, a; *)$, then Def$(\circ)$ becomes:

Definition $(\circ_a)$ $\qquad xy \circ_a \overset{\text{def}}{=} \xi(x, y, a; *)$

and the operation ω can be represented by some term $t(x_1, \ldots x_n; \circ_\omega)$ i. e. it is not necessary to use a new constant, since $\circ_\omega$ already depends on ω. This situation occurs in the case of associative and entropic laws (if for $\xi(x, y; *)$ the term $xa * ay **$ is chosen).

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1) $(x, y) \overset{\text{def}}{=} \{\{x, x\}, \{x, y\}\} = \{\{x\}, \{x, y\}\}$.

3. The converse of the Theorem 1. is given by the following

Theorem 2. *If $\Sigma(*)$ are algebraic laws such that for each Ω-algebra there exists a groupoid $(G, *)$ satisfying the laws $\Sigma(*)$, and Q is isomorphically embedded in $(G, *)$, then there exists a term $\xi(x, y; *)$ such* difficult *the operation $\circ$ defined by Def$(\circ)$ does not satisfy any algebraic law.*

Proof. If each Ω-algebra can be embedded in some groupoid satisfying $\Sigma(*)$, then the same holds for $\circ$-word algebra W generated by some set X and the operation symbol $\circ$ (of arity two). That is, there exists a groupoid $(G, *)$ and a term $\xi(x, y; *)$ such that

$$xy\circ \stackrel{\text{def}}{=} \xi(x, y; *).$$

As W does not satisfy any algebraic law [1], the term $\xi(x, y; *)$ is the required term.

4. By analysing the proofs of the previous theorems it is not hard to see that the assumption that $\Sigma(*)$ are the groupoid laws is not essential. Namely, if $\Sigma(\mathcal{O})$ are the laws with respect to the operators of some set $\mathcal{O}$, but such that:

There exists a term $\xi(x, y; o_1, \ldots, o_n)$, $o_i \in \mathcal{O}$ such that the operation $\circ$ (of arity two) defined by

$$(\mathcal{C}') \qquad xy\circ \stackrel{\text{def}}{=} \xi(x, y; o_1, \ldots, o_n)$$

does not satisfy any algebraic law,

then Theorems 1 and 2 can be extended to the statements about the embeddine of each Ω-algebra in some $\mathcal{O}$-algebra satisfying $\Sigma(\mathcal{O})$, i. e. belonging to thg variety $V_{\mathcal{O}}(\Sigma)$. For example, one primitive class $\Sigma(\mathcal{O})$ satisfying $(\mathcal{C}')$ is the class $\Sigma(\{f\})$ with the law:

$$(x_{11}x_{12}\cdots x_{1n}f\cdots x_{n1}x_{n2}\cdots x_{nn}f)f = (x_{11}x_{21}\cdots x_{n1}f\cdots x_{1n}x_{2n}\cdots x_{nn}f)f.$$

The term of n variables satisfying $(\mathcal{C}')$ is:

$$(x_{11}a\cdots af\cdots aa\cdots x_{nn}f)f$$

which can easily be reduced to a term of the form $\xi(x, y, a; f)$.

The other class $\Sigma(\mathcal{O})$ is $\Sigma(\{f, g\})$ (f of arity one, g of arity two) with the laws:

$$xxg = xf, \qquad xyg = yxg$$

The convenient term $\xi(x, y; f, g)$ is: $xfxygg$. If f, g are the set operator $\{\ \}$ then the term becomes $\{\{x\}, \{x, y\}\}$.

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