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A SINGULAR CONVOLUTION EQUATION IN THE SPACE OF DISTRIBUTIONS

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1. Introduction

The study of singular integral equations in the spirit of generalized functions seems to have orginated for the first time with the papers [1], [2], [5], [6], [8], [9], [10], [11]. The method used is very specific and therefore it is applicable only to convenietly constructed classes of generalized functions.

In the present paper we shall discuss the singular convolution equation (pointed out in [12]) of the type

$$a(t)\,T+\frac{b(t)}{\pi i}\left(T * vp\,\frac{1}{t}\right)=S \tag{1}$$

with respect to unknown distribution T. Here $a(t)$, $b(t)$ are certain infinitely differentiable complex-valued functions defined on R while S is a given distribution in $\mathcal{E}'=\mathcal{E}'(R)$ or $\mathcal{O}_\alpha'=\mathcal{O}_\alpha'(R)$. A role of the distributional convolution process is stressed in [18].

The equation (1) will be solved without technique of an appropriate convolution algebra (if it exists). The basic idea is first to reduce the solution of (1) to the solution of a distributional Hilbert boundary problem for half-planes and thereafter to solve this problem paralleling the classical approach ([7]). The Plemelj formulas extended to distributions in $\mathcal{E}'$ and $\mathcal{O}_\alpha'$ ([12], [14]) are the main tool in proving.

We recall that $\mathcal{E}'$ is the space of distributions with compact support. Let α be any real number. We say (according to a modified definition in [4]) that $\varphi(t)\in\mathcal{O}_\alpha$ if $\varphi(t)\in C^\infty(R)$ and for each nonnegative integer p there exists a constant $M_p>0$ such that $|D^p\varphi(t)|\leq M_p(1+|t|)^\alpha$ for all $t\in R$. A sequence $\{\varphi_n(t)\}$ is said to converge in $\mathcal{O}_\alpha$ to zero if the following are satisfied: (1) each

$\varphi_n(t)\in\mathcal{O}_\alpha$; (2) for each p the sequence $\{D^p\varphi_n(t)\}$ converges uniformly on every compact subset of R to zero; (3) for each p there exists a constant M_p, independent of n such that $|D^p\varphi_n(t)|\leqq M_p(1+|t|)^\alpha$ for all $t\in R$. We then denote $\mathcal{O}_\alpha'$ as the space of all continuous linear functionals on $\mathcal{O}_\alpha$. Also note the important inclusions (proper): $\mathcal{D}\subset\mathcal{O}_\alpha\subset\mathcal{E}$, $\mathcal{E}'\subset\mathcal{O}_\alpha'\subset\mathcal{D}'$ for any $\alpha\in R$ ($\mathcal{D}'$: space of Schwartz distributions). In particular, if $\beta<\alpha$ then $\mathcal{O}_\beta\subset\mathcal{O}_\alpha$ and $\mathcal{O}_\alpha'\subset\mathcal{O}_\beta'$. Finally, the open upper and the open lower half-plane will be denoted by

$$\Delta^+=\{z:Im(z)>0\},\quad \Delta^-=\{z:Im(z)<0\},$$

respectively.

3. Auxiliary results

In preparation for solving the equation (1) let us first recall the Plemelj distributional formulas ([12], [14]): If $T\in\mathcal{E}'$ or $T\in\mathcal{O}'_\alpha$ for $\alpha\geqq-1$ and

$$\hat{T}(z)=\frac{1}{2\pi i}\left\langle T_\tau,\frac{1}{\tau-z}\right\rangle,\quad z\in\Delta^\pm, \tag{2}$$

then the limits $\hat{T}^\pm=\lim\limits_{\varepsilon\to+0}\hat{T}(t\pm i\varepsilon)$ exist in $\mathcal{D}'$ and

$$\hat{T}^+-\hat{T}^-=T, \tag{3}$$

$$\hat{T}^++\hat{T}^-=-\frac{1}{\pi i}\left(T*vp\,\frac{1}{t}\right). \tag{4}$$

If $T\in\mathcal{O}_\alpha'$ for $-1\leqq\alpha<0$, then the formulas (3) and (4) are true in $\mathcal{O}_\alpha'$.

Note that the Cauchy representation $\hat{T}(z)$ of T is a locally (sectionally) holomorphic function with a boundary on the real axis consisting of the support of T that is,

$$\hat{T}(z)=\hat{T}^+(z)\ \text{for}\ z\in\Delta^+,$$

$$\hat{T}(z)=\hat{T}^-(z)\ \text{for}\ z\in\Delta^-.$$

The functions $\hat{T}^+(z)$ and $\hat{T}^-(z)$ are not, in general, the analytic continuation of each other. Also, $\hat{T}(z)=O\left(\frac{1}{|z|}\right)$ as $|z|\to\infty$

Moreover the following information concerning distributional limits of a product with the functions of type $\hat{T}(z)$ will be later useful.

Let $h(z)$ be a locally holomorphic in C (except perhaps at some complex points) cut along the boundary R. Assume that all derivates of $h(z)$ are locally continuous, that is, $\lim\limits_{\varepsilon\to+0}D^p h(t+i\varepsilon)=D^p h^+(t)$ and $\lim\limits_{\varepsilon\to+0}D^p h(t-i\varepsilon)=D^p h^-(t)$ exist for each $p=0, 1, 2,\ldots$ Then

$$\lim_{\varepsilon\to+0}[h(t+i\varepsilon)\,\check{T}(t+i\varepsilon)]=h^+(t)\,\hat{T}^+, \tag{4.1}$$

$$\lim_{\varepsilon\to+0}[h(t-i\varepsilon)\,\hat{T}(t-i\varepsilon)]=h^-(t)\,\hat{T}^- \tag{4.2}$$

in $\mathcal{D}'$.

To establish the above results let us first observe that the function $h^+(t)$ ($h^-(t)$) is necessarily continuous on R. The same is valid for its derivatives. So $h^+(t)$ and $h^-(t)$ belong to $C^\infty(R)$. By definition,

$$\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\hat{T}(t+i\varepsilon)\varphi(t)\,dt=\langle\hat{T}^+,\varphi\rangle$$

for all $\varphi(t)\in\mathfrak{D}$. Let $k(\varepsilon)$ be a function vanishing with ε. Then we may write

$$\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}h(t+i\varepsilon)\,\hat{T}(t+i\varepsilon)\,\varphi(t)\,dt=\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}[h(t+i\varepsilon)-h^+(t)]\,\hat{T}(t+i\varepsilon)\,\varphi(t)\,dt$$

$$+\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}h^+(t)\,\hat{T}(t+i\varepsilon)\,\varphi(t)\,dt=$$

$$=\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}i\varepsilon\,[D^1h^+(t)+k(\varepsilon)]\,\hat{T}(t+i\varepsilon)\,\varphi(t)\,dt+\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\hat{T}(t+i\varepsilon)\,[h^+(t)\,\varphi(t)]\,dt.$$

The limit of the integral involving $D^1h^+(t)$ and $k(\varepsilon)$ is equal to zero. The limit of the last integral is

$$\langle\hat{T}^+,h^+(t)\,\varphi(t)\rangle=\langle h^+(t)\,\hat{T}^+,\varphi(t)\rangle.$$

Likewise, for the second limit it follows

$$\langle\hat{T}^-,h^-(t)\,\varphi(t)\rangle=\langle h^-(t)\,\hat{T}^-,\varphi(t)\rangle.$$

This finishes the proof.

In addition, if the functions $h^+(t)$ and $h^-(t)$ are bounded on R with all their derivatives, then the equalities just established are valid in the $\mathcal{O}_\alpha'$ topology ($-1\leq\alpha<0$). To justify the multiplication of a distribution in $\mathcal{O}_\alpha'$ by an infinitely smooth function, we need the

Lemma 1. *If $h(t)$ is a complex-valued $C^\infty(R)$ — function with the property $|D^p h(t)|\leq A_p$ on $R\,(p=0,1,2,\dots)$ then it is a multiplier for any $\mathcal{O}_\alpha'$, $\alpha\in R$* (here the A_p are constants).

The proof depends on the well-known Leibniz formula for the successive derivatives of a product of two functions in $C^\infty(R)$.

Let α be any fixed number in R. The relation $\varphi(t)\in\mathcal{O}_\alpha$ implies at once $h(t)\,\varphi(t)\in\mathcal{O}_\alpha$. Now suppose that the sequence $\{\varphi_n(t)\}$ converges to zero in $\mathcal{O}_\alpha$ and let $f_n(t)=h(t)\,\varphi_n(t)$. Obviously, $f_n(t)\in\mathcal{O}_\alpha$ for $n=1,2,\dots$ By hypothesis on the functions $\varphi_n(t)$ we have the inequalities

$$|D^p\varphi_n(t)|\leq M_p(1+|t|)^\alpha,$$

where the constants M_p are independent of n and $t\in R$. Since all derivatives of $h(t)$ are bounded on R, using the Leibniz formula it follows that there exist the constants B_p such that

$$|D^pf_n(t)|\leq B_p(1+|t|)^\alpha$$

uniformly with respect to n and $t \in R$ $(p=0, 1, 2, \ldots)$. By reason of the boundedness of $D^p h(t)$, the sequence $\{D^p f_n(t)\}$ converges uniformly to zero on every compact subset of R. Thus the sequence $\{h(t)\varphi_n(t)\}$ converges to zero in $\mathcal{O}_\alpha$. The conditions for the function $h(t)$ to be a multiplier for $\mathcal{O}_\alpha$ are thereby fulfilled. This completes the proof. Now if $T \in \mathcal{O}_\alpha'$, then we define $h(t)\,T$ by $\langle h(t)\,T, \varphi(t)\rangle = \langle T, h(t)\varphi(t)\rangle$ for all $\varphi(t) \in \mathcal{O}_\alpha$. Observe that any $C^\infty(R)$-function is a multiplier for $\mathcal{D}$ and $\mathcal{E}$.

The second important result (after Plemelj distributional formulas) in the proofs of the present statements is a distributional version of a classic theorem of Painlevé concerned with the analytic continuation of holomorphic functions ([16], p. 46). This version adapted for our purpose may be formulated and proved as follows (using the *weak* distributional convergence).

Lemma 2. *Let $f^+(z)$ and $f^-(z)$ be holomorphic functions in the planes Δ^+ and Δ^- with the property $f^\pm(z) = O\left(\frac{1}{|z|}\right)$ as $|z| \to \infty$, respectively. Suppose that these functions separately converge in the $\mathcal{D}'$ topology to the distributional boundary values f^+ and $f^- \in \mathcal{D}'$ such that $\langle (f^+ - f^-), \varphi\rangle = 0$ for all $\varphi(t) \in \mathcal{D}$ whose support lies in some open set $\Omega \subset R$. Then there exists an unique function $f(z)$ that is equal to $f^+(z)$ in Δ^+, $f^-(z)$ in Δ^-, and holomorphic in $\Delta^+ \cup \Omega \cup \Delta^-$.*

Proof. Let Ω be an open interval (the extension to open sets follows immediately since open sets are unions of disjoint open intervals).

First of all we shall show that the distributions f^+, f^- belong to $\mathcal{O}'_{-1}$. In fact, there exist the constants $t_0 > 0$ and $A > 0$ such that for all $|t| > t_0$ the inequalities

$$|f^\pm(t \pm i\varepsilon)| \leq \frac{A}{\sqrt{t^2+\varepsilon^2}} < \frac{A}{|t|} \qquad (\varepsilon > 0)$$

hold. Hence, for all $\varphi(t) \in \mathcal{D}$ with support contained in $\{t : |t| > t_0\}$ it follows

$$|\langle f^\pm, \varphi\rangle| = \lim_{\varepsilon \to +0} \left| \int_{-\infty}^{\infty} f^\pm(t \pm i\varepsilon)\,\varphi(t)\,dt \right| \leq A \int_{-\infty}^{\infty} \frac{|\varphi(t)|}{|t|}\,dt.$$

In other words, the distributions f^+, f^- have the asymptotic bound $|t|^{-1}$. In view of a theorem ([3], p. 54) these distributions can be extended from $\mathcal{D}'$ to $\mathcal{O}_\alpha'$ for any $\alpha < 0$. Thus $f^+, f^- \in \mathcal{O}'_{-1}$. The function $\frac{1}{t-z}$ being an element of $\mathcal{O}_\alpha$ (with respect to t) for any $\alpha \geqq -1$, $Im(z) \neq 0$, the expression

$$\left\langle f_t^+, \frac{1}{t-z} \right\rangle$$

is well-defined and the Cauchy representation of f^+ is given by the equality

$$\frac{1}{2\pi i}\left\langle f_t^+, \frac{1}{t-z}\right\rangle = \frac{1}{2\pi i} \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} \frac{f(t+i\varepsilon)}{t-z}\,dt$$

for any $z \in \Delta^+$. Using the Cauchy integral formula as in [13] we get

$$f^+(z) = \frac{1}{2\pi i}\left\langle f_t^+, \frac{1}{t-z}\right\rangle \quad \text{for } z \in \Delta^+,$$

$$f^+(z) = 0 \quad \text{for } z \in \Delta^-.$$

Similarly, we find

$$f^-(z) = 0 \quad \text{for } z \in \Delta^+,$$

$$f^-(z) = -\frac{1}{2\pi i}\left\langle f_t^-, \frac{1}{t-z}\right\rangle \quad \text{for } z \in \Delta^-.$$

Now if define

$$f(z) = \frac{1}{2\pi i}\left\langle f_t^+, \frac{1}{t-z}\right\rangle - \frac{1}{2\pi i}\left\langle f_t^-, \frac{1}{t-z}\right\rangle, \quad z \in \Delta^\pm,$$

we have

$$f(z) = f^+(z) \quad \text{for } z \in \Delta^+,$$

$$f(z) = f^-(z) \quad \text{for } z \in \Delta^-.$$

To complete the proof we must only show that $f(z)$ is holomorphic on the open interval Ω (by definition $f(z)$ is holomorphic in Δ^+ and Δ^-). Let $a(t)$ be a $C^\infty(R)$—function equal to one on supp $(f^+ - f^-)$, that is, on the complement of Ω, and zero on some arbitrary closed interval $\Omega' \subset \Omega$. We may write

$$f(z) = \frac{1}{2\pi i}\left\langle (f^+ - f^-)_t, \frac{1}{t-z}\right\rangle = \frac{1}{2\pi i}\left\langle (f^+ - f^-)_t, \frac{a(t)}{t-z}\right\rangle.$$

If now z tends to a point $t_0 \in \Omega'$, the then functions

$$\frac{a(t)}{t-z} \text{ converge to } \frac{a(t)}{t-t_0} \text{ in } O_{-1}.$$

Because $f^+ - f^-$ is continuous on O_{-1},

$$\lim_{z \to t_0} f(z) = \frac{1}{2\pi i}\left\langle (f^+ - f^-)_t, \frac{a(t)}{t-t_0}\right\rangle.$$

This shows that $f(z)$ is continuous on Ω'. Hence $f(z)$ is holomorphic on Ω'. Since distance between Ω and Ω' can be made arbitrarily small, $f(z)$ is holomorphic on Ω. Thus $f(z)$ is holomorphic in $\Delta^+ \cup \Omega \cup \Delta^-$ and $f^+(z), f^-(z)$ are analytic continuation of each other. The function $f(z)$ is unique. In fact, let us assume that there exists an other function $f_1(z)$ and define $g(z) = f(z) - f_1(z)$. Supposing that $f(z)$ and $f_1(z)$ were not identical would lead to the conclusion that nonzero function $g(z)$ which converges in the $\mathcal{D}'$ topology could have a distributional boundary value equal to zero in Ω. But this is impossible ([15], Proposition 2).

Consequences: (1) Let the functions $f^+(z)$ and $f^-(z)$ satisfy the conditions of Lemma 2. If $\Omega = R$ and if $f^+(z)$ or $f^-(z)$ has a pole of order m at the complex point $z=a$, then by virtue of generalized Liouville's theorem

$$f(z) = \frac{p_{m-1}(z)}{(z-a)^m},$$

where $p_{m-1}(z)$ is a polynomial of degree $\leqq m-1$; (2) Let the functions $f^+(z)$ and $f^-(z)$ satisfy the conditions of Lemma 2. If $\Omega = R$, then $f(z) \equiv 0$ for $z \in \bar{C}$.

It can be easily verified that the assertion of Lemma 2 together with the previous consequences is true if the $\mathcal{D}'$ topology is replaced with the $\mathcal{O}_\alpha'$ topology $(-1 \leqq \alpha < 0)$. For the proof of the uniqueness we use Proposition 4 in [15]. Clearly, the first part of the proof of Lemma 2 is needless (a complete study about analytic continuation of holomorphic functions in the sense of distributions is exposed in [17]).

At the end of this section we shall solve a distributional boundary problem of Plemelj as a third auxiliary result (in place of the terms "theorem-proof", we shall use the terms "problem-solution").

Problem *1. Let F be a given distribution in $\mathcal{O}_\alpha'$ for $\alpha \geqq -1$. Find a function $f(z)$ locally holomorphic in the plane C cut along the supp $F \cup \{a\}$ and satisfying the boundary condition $f^+ - f^- = F$ on R, where $f^\pm = \lim_{\varepsilon \to +0} f(t \pm i\varepsilon)$ in the $\mathcal{D}'$ topology. One supposes that the complex point a is a pole of order m of $f(z)$ and $f(z) = O\left(\frac{1}{|z|}\right)$ as $z \to \infty$.*

Solution. Consider the function

$$\hat{F}(z) = \frac{1}{2\pi i}\left\langle F_\tau, \frac{1}{\tau - z}\right\rangle. \tag{5}$$

It is locally holomorphic in C cut the supp F and $\hat{F}(z) = O\left(\frac{1}{|z|}\right)$ as $|z| \to \infty$. Now let us define the function $H(z) = f(z) - \hat{F}(z)$ locally holomorphic in C cut along the supp $F \cup \{a\}$. Evidently, $H(z) = O\left(\frac{1}{|z|}\right)$ as $|z| \to \infty$. After a simple calculation we obtain

$$\langle (H^+ - H^-), \varphi\rangle = \langle (f^+ - f^-), \varphi\rangle - \langle \subset \hat{F}^+ - \hat{F}^-), \varphi\rangle$$

for all $\varphi(t) \in \mathcal{D}'$. According to Plemelj formula (3) from (5) it follows

$$\langle (\hat{F}^+ - \hat{F}^-), \varphi\rangle = \langle (f^+ - f^-), \varphi\rangle$$

for all $\varphi(t) \in \mathcal{D}$. Hence $\langle H^+, \varphi\rangle = \langle H^-, \varphi\rangle$ for all $\varphi(t) \in \mathcal{D}$. Since the functions $H^+(z)$ and $H^-(z)$ satisfy the conditions of the consequence (1) of Lemma 2,

$H(z)$ is a rational function in C vanishing at infinity. Consequently, the general solution of the Problem 1 is given by

$$f(z)=\frac{1}{2\pi i}\left\langle F_{\tau}, \frac{1}{\tau-z}\right\rangle+\frac{p_{m-1}(z)}{(z-a)^m}, \tag{5.1}$$

where $p_{m-1}(z)$ is an arbitrary polynomial of degree $\leqq m-1$.

A verification: in view of the Plemelj distributional formulas expressed explicitly we have

$$f^{+}=\frac{F}{2}-\frac{1}{2\pi i}\left(F * vp\frac{1}{t}\right)+\frac{p_{m-1}(t)}{(t-a)^m},$$

$$f^{-}=-\frac{F}{2}\frac{1}{2\pi i}\left(F * vp\frac{1}{t}\right)+\frac{p_{m-1}(t)}{(t-a)^m},$$

in $\mathcal{D}'$ (here the rational function is a regular distribution in $\mathcal{O}_{\alpha}'$ for any $\alpha<0$). Hence $f^{+}-f^{-}=F$.

In Problem 1 let F be a given distribution in $\mathcal{O}_{\alpha}'(-1\leqq\alpha<0)$ and let f^{+}, f^{-} exist in the $\mathcal{O}_{\alpha}'$ topology $(-1\leqq\alpha<0)$. If the other conditions remain the same, it is readily seen that the solution (5.1) is again true. In particular, if $f^{+}=f^{-}$ on R in the sense $\mathcal{O}_{\alpha}'(-1\leqq\alpha<0)$ the Cauchy representation of F is equal to zero and the solution is reduced to the rational function in (5.1). If the function $f(z)$ in Problem 1 is required to be locally holomorphic in C cut only along the supp F, then the solution (5.1) is given by the Cauchy representation of F.

3. Hilbert transform.

Before stating a solution of (1) it may be of interest to solve separately the following simplest convolution equation of the type under consideration:

Let S be a given distribution in $\mathcal{E}'$. Find the unknown distribution T such that

$$\frac{1}{\pi i}\left(T * vp\frac{1}{t}\right)=S. \tag{6}$$

Solution. First of all recall that $vp\dfrac{1}{t}$ is a distribution in $\mathcal{O}_{\alpha}'$ for any $\alpha<0$ (with the support equal to the whole R). Let T be a distribution in $\mathcal{D}'$. According to the definition of the convolution we may write

$$\left\langle T_t * vp\frac{1}{t}, \varphi\right\rangle=\left\langle T_t, \left\langle vp\frac{1}{\tau-t}, \varphi(\tau)\right\rangle\right\rangle=\left\langle T_t, \int_{-\infty}^{\infty}\frac{\varphi(\tau)}{\tau-t}\,d\tau\right\rangle= \tag{7}$$

$=2\pi\langle T_t, \hat{\varphi}(t)\rangle$ for all $\varphi(t)\in\mathcal{D}$. Since the singular Cauchy integral $\hat{\varphi}(t)$ is not an element of $\mathcal{D}$, the last expression in (7) is not defined ($\hat{\varphi}(t)\in\mathcal{O}_{-1}'$ for all $\varphi(t)\in\mathcal{D}$). Thus the unknown distribution T belongs to a subspace of $\mathcal{D}'$. Suppose $T\in\mathcal{E}'$. Now the convolution (7) is well-defined in $\mathcal{D}'$ but this hypothesis relative to the equation (6) leads to a contradiction. Indeed, the equality $S=0$ on R — supp S is at the same time valid on $\Omega=(R - \text{supp } S)\cap(R - \text{supp } T)$. In view of the formula (4) the relation $S=0$ implies $\hat{T}^+ = -\hat{T}^-$ on Ω. On the other hand the assumption $T\in\mathcal{E}'$ implies $\hat{T}^+ = \hat{T}^-$ on Ω. A contradiction is reached. This enables us to conclude that T lies in an intermediate space $\mathcal{O}_\alpha'$. Suppose $T\in\mathcal{O}_\alpha'$ with an index $\alpha\geqq 0$. In this case it is not possible to define the convolution (7) on the space $\mathcal{O}_\alpha$. Clearly, it is well-defined in $\mathcal{D}'$ but then the restriction of T is acting. Hence the distribution T is not contained in any one $\mathcal{O}_\alpha'$ with $\alpha\geqq 0$. Finally suppose $T\in\mathcal{O}_\alpha'$ with an index $\alpha\in[-1, 0)$. Then according to the formula (4) the first expression in (7) is well-defined for all $\varphi(t)\in\mathcal{O}_\alpha$. The discussion indicates that T lies in the spaces $\mathcal{O}_\alpha'$, $-1\leqq\alpha<0$. In order to find a such T we introduce once more the Cauchy representation

$$\hat{T}(z)=\frac{1}{2\pi i}\left\langle T_\tau, \frac{1}{\tau-z}\right\rangle, \quad z\in\Delta^\pm. \tag{8}$$

In virtue of (4) the equation (6) may be written in the form

$$-\hat{T}^+ - \hat{T}^- = S. \tag{9}$$

At this point consider a second locally holomorphic function $\hat{S}(z)$ defined by

$$\hat{S}^+(z) = -\hat{T}^+(z) \quad \text{for} \quad z\in\Delta^+,$$

$$\hat{S}^-(z) = \hat{T}^-(z) \quad \text{for} \quad z\in\Delta^-.$$

Then the boundary relation (9) becomes $\hat{S}^+ - \hat{S}^- = S$ and in view of the solution of Problem 1 we have

$$\hat{S}(z)=\frac{1}{2\pi i}\left\langle S_\tau, \frac{1}{\tau-z}\right\rangle, \quad z\in\Delta^\pm.$$

On the other hand, using the Plemelj distributional formulas repeatedly it follows $T=\hat{T}^+ - \hat{T}^- = -\hat{S}^+ - \hat{S}^-$ and

$$T=\frac{1}{\pi i}\left(S * vp\,\frac{1}{t}\right). \tag{10}$$

Since S operates on the singular Cauchy integral $\hat{\varphi}(t)\in C^\infty(R)$ with density function $\varphi(t)\in\mathcal{O}_\alpha$ for each $\alpha<0$, we see now from (10) that T is a distribution in $\mathcal{O}_\alpha'$ for any $\alpha<0$ (the fact that $\hat{\varphi}(t)$ is an element of $C^\infty(R)$ follows from classical Plemelj formulas for derivatives of $\hat{\varphi}(z)$).

If S is a given distribution in $\mathcal{O}_\alpha'$ for any $\alpha\in[-1, 0)$, it is manifest that T is determined again by (10) and belongs to the same $\mathcal{O}_\alpha'$ (this implies $T\in\mathcal{O}_\beta'$ for all $\beta<\alpha$). Thus (10) follows from (6). Conversely, if T is given in $\mathcal{O}_\alpha'$ for any $\alpha\in[-1, 0)$ in the same way (6) follows from (10). Hence (10) is the unique solution of the equation (6). In other words, the convolution equations (6) and (10) in this case are equivalent.

A verification: substituting the solution (10) in (6) we obtain

$$\frac{1}{\pi i}\left[\frac{1}{\pi i}\left(S * vp\frac{1}{t}\right) * vp\frac{1}{t}\right] = -\frac{1}{\pi^2}\left[S * \left(vp\frac{1}{t} * vp\frac{1}{t}\right)\right] =$$

$$-\frac{1}{\pi^2}[S * (-\pi^2\delta)] = S * \delta = S \text{ (δ being Dirac distribution).}$$

4. Main result

In this section the general solution of (1) in the closed form will be given.

Problem 2. *Let $a(t)$ and $b(t)$ be given complex-valued functions in $C^\infty(R)$ such that $a(t) \pm b(t) \neq 0$ on R, $D^p a(t) = O(1)$ and $D^p b(t) = O(1)$ as $|t| \to \infty$ $(p = 0, 1, 2, \ldots)$. In addition, suppose that the quotient $\frac{a(t)+b(t)}{a(t)-b(t)}$ tends to one the same value different from zero as $t \to \pm\infty$ and satisfies a Hölder (H) condition at infinity. Let S be a given distribution in $\mathcal{S}'$. Find the distribution $T \in O_\alpha'$ with $-1 \leq \alpha < 0$ such that*

$$a(t)T + \frac{b(t)}{\pi i}\left(T * vp\frac{1}{t}\right) = S. \tag{11}$$

Solution. Assume that T satisfies the equation (11). Acting on testing functions $\frac{1}{\tau - z}$ the distribution T generates the function

$$\hat{T}(z) = \frac{1}{2\pi i}\left\langle T_\tau, \frac{1}{\tau - z}\right\rangle, \quad z = \Delta^\pm. \tag{12}$$

With the aid of the Plemelj distributional formulas, we reduce (11) to the Hilbert boundary problem

$$[a(t) - b(t)]\hat{T}^+ - [a(t) + b(t)]\hat{T}^- = S. \tag{12.1}$$

Since T satisfies (11) the function $\hat{T}(z)$ defined by (12) must be a solution of (12.1) which vanishes at infinity. Conversely, let the locally holomorphic function $\hat{T}(z)$ (with a boundary on R) vanishing as $\frac{1}{|z|}$ when $|z| \to \infty$ be a solution of the problem (12.1). Define T by setting $T = \hat{T}^+ - \hat{T}^-$. Solving this boundary problem (Problem 1), the function $\hat{T}(z)$ may be written in the form (12), and hence the formula

$$-\frac{1}{\pi i}\left(T * vp\frac{1}{t}\right) = \hat{T}^+ + \hat{T}^-$$

will be also true. Since $\hat{T}(z)$ is a solution of (12.1) it follows that T is a solution of (11). We summarize: a solution of (11) is equivalent to the solution of the problem (12.1) with the supplementary condition $\hat{T}(z) = O\left(\frac{1}{|z|}\right)$ as $|z| \to \infty$. Consequently, the problem

$$\hat{T}^+ = \frac{a(t)+b(t)}{a(t)-b(t)}\,\hat{T}^- + \frac{S}{a(t)-b(t)} \tag{13}$$

has now to be solved. Observe here that the given quotient is a multiplier for $\hat{T}^-$. Let $G(t) = \frac{a(t)+b(t)}{a(t)-b(t)}$. *Suppose that the index* $\lambda = \frac{1}{2\pi i}\{\log G(t)\}_{\bar{R}}$ *of the problem is* $\geqq 0$. Here the brace denotes the total increment of $\log G(t)$ on the whole R, that is, when t goes on R from $-\infty$ to $+\infty$.

Let $X^+(z) = \exp \Gamma^+(z)$ and $X^-(z) = \left(\frac{z+i}{z-i}\right)^\lambda \exp \Gamma^-(z)$ be canonical functions of the problem, where

$$\Gamma(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \log\left[\left(\frac{\tau+i}{\tau-i}\right)^\lambda G(\tau)\right]\frac{d\tau}{\tau-z}, \quad z \in \Delta^\pm. \tag{14}$$

The first Plemelj classical formula applied to $\Gamma(z)$ implies

$$\frac{X^+(t)}{X^-(t)} = \frac{a(t)+b(t)}{a(t)-b(t)}.$$

The *substitution* in (13) gives

$$\frac{\hat{T}^+}{X^+(t)} = \frac{\hat{T}^-}{X^-(t)} + \frac{S}{X^+(t)\,[a(t)-b(t)]}. \tag{15}$$

Now if we assume that each derivative of the integrand (without Cauchy kernel) in (14) has a property H at infinity, the functions $X^+(t)$ and $X^-(t)$ (bounded and different trom zero on R) become the elements of $C^\infty(R)$. Moreover, all derivatives of these functions are bounded on R. Hence, by Lemma 1 the functions $X^+(t)$ and $X^-(t)$ are multipliers for $\mathcal{O}_\alpha$ with any $\alpha \in R$. The first and the second term in (15) are members of $\mathcal{O}_\alpha'$ $(-1 \leqq \alpha < 0)$. The third term belongs again to $\mathcal{E}' \subset \mathcal{O}_\alpha'$ and its support is precisely supp S.

At this point we introduce the locally holomorphic function

$$\hat{S}(z) = \frac{1}{2\pi i}\left\langle \frac{S}{X^+(\tau)\,[a(\tau)-b(\tau)]},\ \frac{1}{\tau-z}\right\rangle, \quad z \in \Delta^\pm.$$

By the first Plemelj distributional formula applied to $\hat{S}(z)$, the boundary relation (15) may be written in the form

$$\frac{\hat{T}^+}{X^+(t)} - \hat{S}^+ = \frac{\hat{T}^-}{X^-(t)} - \hat{S}^-, \tag{16}$$

where $\hat{S}^{\pm}=\lim_{\varepsilon\to+0}\hat{S}(t\pm i\varepsilon)$ in $\mathcal{O}_\alpha'$, $-1\leqq\alpha<0$. In order to deduce from (16) the unknown function $\hat{T}(z)$, define a new locally holomorphic function

$$\hat{W}(z)=\frac{\hat{T}(z)}{X(z)}-\hat{S}(z) \tag{17}$$

in C cut along the $R\cup\{-i\}$, where $X(z)=X^+(z)$ for $z\in\Delta^+$, $X(z)=X^-(z)$ for $z\in\Delta^-$. Let us observe that the function $\hat{W}(z)$ has in the domain $C-R$ as unique singularity a pole of order λ at $z=-i$. Since the function $X(z)$ is bounded and $\hat{T}(z)$, $\hat{S}(z)$ vanish as $\frac{1}{|z|}$ when $|z|$ tends to infinity, we have $\hat{W}(z)=0\left(\frac{1}{|z|}\right)$ when $|z|\to\infty$. In addition, using the results (4.1) and (4.2) we see that the limits $\hat{W}^{\pm}=\lim_{\varepsilon\to+0}\hat{W}(t\pm i\varepsilon)$ exist in $\mathcal{O}_\alpha'$ $(-1\leqq\alpha<0)$ and

$$\hat{W}^+=\frac{\hat{T}^+}{X^+(t)}-\hat{S}^+,\quad \hat{W}^-=\frac{\hat{T}^-}{X^-(t)}-\hat{S}^-.$$

But the relation (16) shows that $W^+=W^-$. Hence, from the solution of Problem 1 formulated in the sense of the $\mathcal{O}_\alpha'$ topology $(-1\leqq\alpha<0)$ we infer that

$$\hat{W}(z)=\frac{P_{\lambda-1}(z)}{(z+i)^\lambda}, \tag{18}$$

where $P_{\lambda-1}(z)$ is an arbitrary polynomial of degree $\leqq\lambda-1$. Now comparing (18) with (17) we get the solution of the problem (13):

$$\hat{T}(z)=X(z)\left\{\frac{1}{2\pi i}\left\langle\frac{S}{X^+(\tau)[a(\tau)-b(\tau)]},\ \frac{1}{\tau-z}\right\rangle+\frac{P_{\lambda-1}(z)}{(z+i)^\lambda}\right\},\quad z\in\Delta^\pm.$$

For $\lambda\leqq0$ it is necessary to put $P_{\lambda-1}(z)\equiv0$. *If* $\lambda<0$ *the problem* (13) *has no solution* (since this hypothesis implies the meromorphy of the function $\hat{T}^-(z)$).

We are now in a position to derive effectively the unknown distribution $T=\hat{T}^+-\hat{T}^-$. Applying the Plemelj distributional formulas on the found function $\hat{T}(z)$ we get:

$$\hat{T}^+=X^+(t)\left\{\frac{S}{2X^+(t)[a(t)-b(t)]}-\frac{1}{2\pi i}\left(\frac{S}{X^+(t)[a(t)-b(t)]}*vp\frac{1}{t}\right)+\frac{P_{\lambda-1}(t)}{(t+i)^\lambda}\right\},$$

$$\hat{T}^-=X^-(t)\left\{-\frac{S}{2X^+(t)[a(t)-b(t)]}-\frac{1}{2\pi i}\left(\frac{S}{X^+(t)[a(t)-b(t)]}*vp\frac{1}{t}\right)+\frac{P_{\lambda-1}(t)}{(t+i)^\lambda}\right\},$$

in $\mathcal{O}_\alpha'$ with $-1\leqq\alpha<0$.
Consequently,

$$T=\frac{X^+(t)+X^-(t)}{2X^+(t)[a(t)-b(t)]}S-\frac{X^+(t)-X^-(t)}{2\pi i}\left(\frac{S}{X^+(t)[a(t)-b(t)]}*vp\frac{1}{t}\right)+[X^+(t)-X^-(t)]\frac{P_{\lambda-1}(t)}{(t+i)^\lambda}. \tag{19}$$

However, the first term in (19) is a distribution in $\mathcal{O}_\alpha'$ for any $\alpha \in R$. Also, the second and third term are distributions in $\mathcal{O}_\alpha'$ for any $\alpha < 0$. This implies $T \in \mathcal{O}_\alpha'$ for any $\alpha < 0$.

Now assume that the given distribution S in Problem 2 is an element in $\mathcal{O}_\alpha'$ *with* $-1 \leqq \alpha < 0$. If the other conditions are not altered, the solution of (11) is determined again by (19). Here the distribution convoluted with $vp \frac{1}{t}$ is an element of $\mathcal{O}_\alpha'$ and all terms of the right side are distributions which operate on the testing space $\mathcal{O}_\alpha$. Hence $T \in \mathcal{O}_\beta'$ for any $\beta \leqq \alpha$.

If we set $a(t) \equiv 0$ and $b(t) \equiv 1$ in (11), then the equation takes the form (6). In this case $G(t) \equiv -1$ on R, $\lambda = 0$, and one may assume $\log G(t) = \pi i$. Therefore

$$\Gamma(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\pi i}{\tau - z} d\tau = +\frac{\pi i}{2} \text{ for } z \in \Delta^+, \quad = -\frac{\pi i}{2} \text{ for } z \in \Delta^-.$$

By the definition

$$X^+(z) = \exp \Gamma^+(z) = +i \text{ for } z \in \Delta^+,$$

$$X^-(z) = \exp \Gamma^-(z) = -i \text{ for } z \in \Delta^-.$$

Thus by (19) we obtain again (10).

As an example the equation

$$2T + \frac{1}{\pi i}\left(T * vp \frac{1}{t}\right) = \delta$$

will be solved. It is evident that $a(t) \equiv 2$, $b(t) \equiv 1$. Hence $G(t) \equiv 3$ on R and $\lambda = 0$. Thus $X^+(z) = \sqrt{3}$, $X^-(z) = \frac{1}{\sqrt{3}}$. Substituting $X^+(t) = \sqrt{3}$, $X^-(t) = \frac{1}{\sqrt{3}}$ in (19), after a short computation we get the solution

$$T = \frac{2}{3}\delta - \frac{1}{3\pi i} vp \frac{1}{t}.$$

Problem 2 can be given under the following stronger conditions: (1) $a(t) \pm b(t)$ is different from zero on R including the point $t = \infty$ (instead of $a(t) \pm b(t) \neq 0$ on R); (2) the derivatives of the functions $a(t)$ and $b(t)$ satisfy a Hölder property at infinity (instead of order relation $O(1)$). In this case it is easy to see that the hypotheses of Problem 2 on the function $G(t) = \frac{a(t) + b(t)}{a(t) - b(t)}$ are satisfied. Also, the additional assumption (made in the course of proving) on the integrand in (14) is satisfied. The argument that the functions $X^+(t)$, $X^-(t)$ exist and belong to $C^\infty(R)$ follows from the Plemelj formulas for the derivatives of $\Gamma(z)$. Consequently, the solution (19) holds again.

REFERENCES

[1] S. Abdykalykov, *A certain class of convolution type dual integral equation in a space of generalized functions* (Russian), Studies in Integrodifferential Equations in Kirgizia n⁰ 4, 129—135, Izdat. „Ilim" 1967.

[2] S. Abdykalykov, *A singular integral equation with shift in a space of generalized functions* (Russian), Izv. Akad. Nauk Kirgiz. SSR (1967), № 3, 26—31.

[3] H. J. Bremermann *Distributions, complex variables, and Fourier transforms*, Addison-Wesley, New York, 1965.

[4] R. D. Carmichael. *Representations of distributions in O_α' as boundary values of functions in tube domains*, Glasnik Mat. 7 (1972), 213—228.

[5] V. B. Dybin, *An exceptional case of integral equations of convolution type in the class of generalized functions* (Russian), Dokl. AN SSSR, 161 (1965). 753—756.

[6[V. B. Dybin, N. K. Karapetjanc, *Integral equations of the convolution type in a class of generalized functions* (Russian), Sibirsk. Mat. Ž. 7 (1966), 531—545.

[7] F. D. Gahov, *Boundary value problems* (Russian), Fizmatgiz, Moscov, 1958.

[8] M. I. Haikin, *The Wiener-Hopf equation in space of ordinary and generalized functions* (Russian), Izv. Vysš. Učebn. Zavod. Matematika (1967), 10 (65), 83—91.

[9] A. E. Kosulin, *One-dimensional singular equation in generalized functions* (Russian), Dokl. Akad. Nauk SSSR, 163 (1965), 1054—1057.

[10] A. E. Kosulin, *One-dimensional singular equations in generalized functions* (Russian), Mat. Sbornik (N. S.), 70 (112) (1966), 180—197.

[11] A. E. Kosulin, *One-dimensional singular equation in generalized functions. The case of an open contour* (Russian), Vestnik Leningrad. Univ., 26 (1966), № 7, 157—160.

[12] D. Mitrović *The Plemelj distributional formulas*, Bull. Amer. Math. Soc. 77 (1971), 562—563.

[13] D. Mitrović, *A note on the distributional boundary values of analytic functions*, Mat. vesnik 8 (23) (1971), 209—211.

[14] D. Mitrović, *On a decomposition of distributions*, Mathematica balkanica 2 (1972), 156—160.

[15] D. Mitrović, *Une remarque sur les valeurs au bord des fonctions holomorphes*, Mathematica balkanica 3 (1973), 363—367.

[16] P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Chelsea Publishing Company, New York, 1974.

[17] W. Rudin, *Lectures on the edge-of-the-wedge theorem*, Amer. Math. Soc., Providence, R. I., 1971.

[18] A. H. Zemanian, *Distributions theory and transform analysis*, Mc Graw-Hill, New York, 1965.

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