PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvell série, tome 21 (35), 1977, pp. 145–150

C-COMPLETE SYSTEMS IN NORMED SPACES

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(Communicated August 30, 1975)

1. Introduction

It is a known fact that an orthonormal sequence (e_i) in a Hilbert space is complete if and only if it is linearly dense in it. In that case it is also Schauder's basis in it. In incomplete inner-product spaces the situation is more complex. That is, a complete orthonormal sequence in an incomplete inner-product space is not necessarily linearly dense nor a basis. M. Golomb has introduced in [1] a concept of C-complete sequence as an equivalent for the concept of linearly dense sequence in the incomplete inner-product space.

That paper ([1]) inspired us to write this article, in which we obtain similar results in incomplete normed spaces. We also found certain necessary and sufficient conditions for an orthonormal sequence to be a Schauder's basis in, so called, semi-inner product space and some generalizations of Parseval's equality and Minkowski's inequality.

2. Orthonormal systems in semi-inner product spaces

Let X be a normed space over the field Φ. (Φ is the field of complexe numbers C or the field of real numbers R). Let $p>1$ be a fixed real number. Let $[.,.]$ stand for a generalized semi-inner product over X^2 (s. i. p. g.) whose index is p and which satisfies conditions:

$$(\forall x \in X)\ ([x, x]=\|x\|^p),$$

$$(\forall x, y \in X)\ (\forall \lambda \in \Phi)\ ([x, \lambda y]=|\lambda|^{p-2}\bar{\lambda}[x, y]).$$

Such s. i. p. g. always exists ([3]). If $p=2$ then $[.,.]$ is semi-inner product (s. i. p.) which is a generalization of inner product $(.,.)$. The space X with an s. i. p. g. $[.,.]$ we denote as s. i. p. s. g. Besides, in this paper, X^* is the dual space of X, S is the unit sphere of X and N is the set of natural numbers.

The following definitions are essential for this paper.

Definition 1. *A vector system* $(e_i)_{i\in I}$ *is orthonormal in s. i. p. s. g.* X *if and only if*

$$(\forall\, i, j\in I)\ ([e_i, e_j]=\delta_{ij}),$$

where δ_{ij} *is Kronecker's delta.*

It can be easily seen that each s. i. p. s. g. possesses at least one orthonormal system and that vectors of an orthonormal system are linearly independent.

Definition 2. *Let* $E=(e_i)_{i\in I}$ *be an orthonormal system in s. i. p. s. g.* X. $\hat{x}_i=[x, e_i]$ *are left Fourier's coefficients and* $\check{x}_i=[e_i, x]$ *are right Fourier's coefficients of the vector* x *relative to the system* E.

Definition 3. *A system* $(e_i)_{i\in I}$ *is complete in s. i. p. s. g.* X *if*

$$((\forall\, i\in I)\ (\hat{x}_i=0))\Rightarrow x=0.$$

Definition 4. *Orthonormal sequence* (e_i) *is C-regular in s. i. p. s. g.* X *if for all* $x\in S$, *the sequence* $\sum_1^n \hat{x}_i e_i$ *is a Cauchy sequence in* X.

In [4], the existence of orthonormal systems which fulfil the conditions of definitions 3. and 4., their cardinality, and necessary and sufficient conditions, for such a system to be a Schauder's basis in a Banach space, are discussed. It was shown that in l^p space with s. i. p. g.

$$(\forall\, x=(x_i),\ y=(y_i)\in l^p)\ ([x, y]=\sum_1^\infty x_i\,|y_i|^{p-1}\operatorname{sgn} y_i), \tag{1}$$

the vector sequence

$$e_n=(\underbrace{0, 0, \ldots, 0}_{n-1}, 1, 0, \ldots)\ (n=1, 2, \ldots) \tag{2}$$

satisfies condition of definitions 1., 3. and 4. If K is the set of all vectors in l^p which have finitely many coordinates different from zero, then K is incomplete s. i. p. s. g. relative to s. i. p. g. (1) and the sequence (2) satisfies conditions of definitions 1., 3. and 4. in K.

Also, in $L^p_{(0,1)}$ the Haar's sequence of functions

$$\varphi_0=1.$$

$$\varphi_{01},$$

$$\varphi_{11},\ \varphi_{12},$$

$$\ldots\ldots\ldots$$

$$\varphi_{n1},\ \varphi_{n2}, \ldots, \varphi_{n2^n},$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\text{where } \varphi_{ni}=\begin{cases} 2^{\frac{n}{p}}, & \frac{i-1}{2^n}<t<\frac{2i-1}{2^{n+1}} \\ -2^{\frac{n}{p}}, & \frac{2i-1}{2^{n+1}}<t<\frac{i}{2^n} \\ 0, & t\notin\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right] \end{cases} \qquad (n=0, 1, 2, \ldots; \ i=1, 2, \ldots 2^n),$$

is orthonormal sequence with properties of definitions 3. and 4. who are relative to s. i. p. g.

$$(\forall\, x, y\in L^p_{(0,1)})\ \left([x, y]=\int_0^1 x\,|y|^{p-1}\operatorname{sgu} y\,dt\right).$$

3. C-Complete orthonormal systems

Definition 5. *A sequence (x_i) in s. i. p. s. g. X is C-complete if each Cauchy sequence (y_n) in X for which*

$$(\forall\, i\in N)\ (\lim_{n\to\infty}[y_n, x_i]=0)$$

is a null sequence, i. e., $\lim\limits_{n\to\infty}\|y_n\|=0$.

The sequence (2) is C-complete in K.

Theorem 1. *A sequence (x_i) in a complete s. i. p. s. g. X is C-complete if and only if it is complete in X.*

Proof. C-completeness implies completeness also in incomplete space. Indeed, if (x_i) is C-complete and if $x\in X$ such that $(\forall\, i\in N)\ ([x, x_i]=0)$ then, the sequence $y_n=x$ is a Cauchy sequence for which

$$(\forall\, i\in N)\ (\lim_{n\to\infty}[y_n, x_i]=[x, x_i]=0).$$

Therefrom, $\lim\limits_{n\to\infty}\|y_n\|=\|x\|=0$.

Conversely, let (x_i) be complete and (y_n) a Cauchy sequence for which

$$(\forall\, i\in N)\ (\lim_{n\to\infty}[y_n, x_i]=0).$$

Since X is complete, there is a $y\in X$ such that $y=\lim\limits_{n\to\infty} y_n$. Since $[.,.]$ is linear and bounded functional in first argument, it follows that

$$(\forall\, i\in N)\ (\lim_{n\to\infty}[y_n, x_i]=[y, x_i]=0).$$

Because of completeness of the sequence (x_i), $y=0$, i. e.,

$$\lim_{n\to\infty}\|y_n\|=\|y\|=0.$$

Theorem 2. 1) *If an orthonormal sequence (e_i) is C-regular and C-complete in X, then it is linearly dense in X.*

2) *If an orthonormal sequence* (e_i) *is linearly dense in* X, *and if the sequence* $f_i=[\cdot, e_i]$ *is linearly dense in* X^*, *then the sequence* (e_i) *is C-complete in* X.

Proof. 1) If a sequence (e_i) is C-regular then the sequence $x_n=x-\sum_1^n \hat{x}_i e_i$ is a Cauchy sequence, as

$$\| x_n-x_m \| = \| \sum_{n+1}^{m} \hat{x}_i e_i \| = \| x \| \ \| \sum_{n+1}^{m} \hat{y}_i e_i \| ,$$

where $x=\| x \| y$ and $y=S$. Moreover,

$$[x_n, e_i]=\begin{cases} 0, & i\leqslant n \\ [x, e_i], & i>n \end{cases}$$

and, therefore,

$$(\forall i\in N)\ (\lim_{n\to\infty} [x_n, e_i]=0).$$

Since (e_i) is C-complete we have $\lim_{n\to\infty} \| x_n \| =0$. Therefore

$$(3) \qquad (\forall \varepsilon>0)\ (\exists n_0\in N)\ (\forall n\geqslant n_0)\ (\| x-\sum_1^n \hat{x}_i e_i \| <\varepsilon),$$

which proves that the sequence (e_i) is linearly dense in X.

2) Let (y_n) be a Cauchy sequence in X such that

$$(4) \qquad (\forall i\in N)\ (\lim_{n\to\infty} [y_n, e_i]=0).$$

(y_n) is bounded. Since (f_i) is linearly dense in X^*, according to theorem 2 p. 184 [2] and (4), we have

$$(\forall x\in X)\ (\lim_{n\to\infty} [y_n, x]=[0, x]=0).$$

Let ε be arbitrary small positive number and $n_0\in N$ such that, for $m>n\geqslant n_0$, $\| y_m-y_n \| <\varepsilon M^{1-p}$ where $M=\sup_n \| y_n \|$. For all $x\in X$, such that $\| x \| \leqslant M$, we have

$$| [y_m, x]-[y_n, x] | = | [y_m-y_n, x] | \leqslant \| y_m-y_n \| \ \| x \|^{p-1} \leqslant \varepsilon M^{1-p} M^{p-1}=\varepsilon.$$

Consequently, the sequence $[y_n, x]$ uniformly converges to 0 for $\| x \| \leqslant M$. Now, let $n'\in N$ be such that $| [y_{n'}, x] | <\varepsilon$. If we set $x=y_{n'}$ then $[y_{n'}, y_{n'}]<\varepsilon$ or $\| y_{n'} \|^p<\varepsilon$. This inequality shows that $\lim_{n\to\infty} \| y_n \| =0$, and the proof is completed.

Remark 1. If an orthonormal sequence (e_i) is linearly dense in an inner product space $(X, (\cdot, \cdot))$, then the sequence $f_i=(\cdot, e_i)$ is linearly dense in X^*. (Remark 1. and Remark 2., p. 178 [2]).

R e m a r k 2. Theorem 2. is a generalization of the main result (Theorem) in [1]. Indeed, each orthonormal sequence (e_i) in an inner product space $(X, (.,.))$ is C-regular in $(X, (.,.))$ and C-completeness of the sequence (x_i) is clearly equivalent to C-completeness of the sequence obtained from (x_i) by orthonormalization.

R e m a r k 3. The proof of theorem 2 is different from the proof of Theorem [1].

T h e o r e m 3. *Let the sequence (e_i) be orthonormal in s. i. p. s. g. X and the sequence $f_i=[.,e_i]$ be linearly dense in X^*. Then the sequence (e_i) is a Schauder's basis if and only if the sequence (e_i) is C-complete and C-regular in X.*

P r o o f. If (e_i) is C-complete and C-regular then (3) holds, i. e.

$$(\forall\, x\in X)\ (x=\lim_{n\to\infty}\sum_1^n \hat{x}_i e_i=\sum_1^\infty \hat{x}_i e_i),$$

which proves that (e_i) is a Schauder's basis in X. Conversely, if (e_i) is a Schauder's basis then there exists a unique sequence $(\varphi_i)\subset X^*$ such that

$$(\forall\, x\in X)\ (x=\sum_1^\infty \varphi_i(x)\, e_i).$$

In that case $(\forall\, k\in N)$ $(\varphi_k(x)=\hat{x}_k)$ therefore $x=\sum_1^\infty \hat{x}_k e_i$. That means that the sequence $\sum_1^n \hat{x}_i e_i$ converges in X, which shows that the sequence (e_i) is C-regular and linearly dense in X. By part 2) of Theorem 2, it is C-complete.

C o r o l l a r y. *If one of the following two conditions is satisfied:*

1. *(e_i) is orthonormal, C-regular and C-complete in X;*

2. *(e_i) is orthonormal, C-regular and linearly dense in X and $f_i=[.,e_i]$ is linearly dense in X^*, then*

$$(\forall\, x\in X)\ (x=\sum_1^\infty \hat{x}_i e_i), \tag{5}$$

$$(\forall\, x, y\in X)\ ([x,y]=\sum_1^\infty \hat{x}_i \check{y}_i), \tag{6}$$

$$(\forall\, x, y\in X)\ ((\sum_1^\infty [x+y, e_i]\,[e_i, x+y])^{1/p}\leqslant(\sum_1^\infty \hat{x}_i \check{x}_i)^{1/p}+(\sum_1^\infty \hat{y}_i \check{y}_i)^{1/p}). \tag{7}$$

The equality (6) is a result of representation (5) and property of linearity and continuity of s. i. p. g. $[\cdot, \cdot]$ in the first argument. Otherwise, if $[\cdot, \cdot]$ is an inner product, it becomes Parseval's equality. The inequality (7) follows from (6), the representation $\| x \| = [x, x]^{1/p}$ and two inequality $\| x+y \| \leqslant \| x \| + \| y \|$. If $X = l^p$ with s. i. p. g. (1), then Minkowski's inequality is obtained

$$(\sum_{1}^{\infty} | x_i + y_i |^p)^{1/p} \leqslant (\sum_{1}^{\infty} | x_i |^p)^{1/p} + (\sum_{1}^{\infty} | y_i |^p)^{1/p}.$$

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