PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série. tome 21 (35), 1977, pp. 141—144

A NOTE ON ELEMENTARY END EXTENSION

Žarko Mijajlović

(Received March 4, 1976)

1. Let R be a binary relation symbol in a countable language L and $\mathfrak{A}$ a countable model for L. Let us define $R_a^{\mathfrak{A}} = \{x \in A \mid \mathfrak{A} \models R(x, a)\}$ for every element $a \in A$. We shall write R_a instead of $R_a^{\mathfrak{A}}$ if there is no ambiguity. A model $\mathfrak{B}$ is an elementary R-end extension of $\mathfrak{A}$ if and only if $\mathfrak{A} < \mathfrak{B}$ (i.e. if $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$), $A \neq B$ and for every $a \in A$ $R_a^{\mathfrak{A}} = R_a^{\mathfrak{B}}$, in which case we shall write $\mathfrak{A} <_{eR} \mathfrak{B}$. Our aim is to give some sufficient conditions for R in order to $\mathfrak{A}$ has an elementary R-end extension. In fact, the work in this paper is related to the proof of the theorem 2.2.18 [2], which says that every countable model of ZF has an elementary end extension. It will appear that a model with a relation R has an elementary end extension under weaker assumption for R, instead of R is $\in$ in ZF.

In the following we need the omitting types theorem:

Theorem 1 (A. Ehrenfeucht) *Let T be a consistent theory in a countable L, and for each $n \in \omega$ let $\Sigma_n(x_1, \ldots, x_{k_n})$ be a set of formulas in k_n variables. If T locally omits each Σ_n, then T has a countable model which omits each Σ_n.*

We say that T localy omits $\Sigma(x_1, \ldots, x_n)$ if for every formula $\varphi(x_1, \ldots, x_n)$ which is consistent with T, there exits $\sigma \in \Sigma$ such that $\varphi \wedge \neg \sigma$ is consistent with T. The proof and applications of the above theorem can be found for example in [2].

2. Definition. A relation R is regular in $\mathfrak{A}$ if and only if the following holds in $\mathfrak{A}$.

C.1. For every $a \in A$ $R_a \neq A$.

C.2. For all $a, b \in A$ there is $c \in A$ such that $a, b \in R_c$.

For a formula $\varphi(x, y)$ in two free variables and $x \in A$, let $\varphi_x^{\mathfrak{A}} = \{y \in A \mid \mathfrak{A} \models \varphi(x, y)\}$.

C.3. Let $a \in A$. If for every $x \in R_a$ there is $b \in A$ such that $\varphi_x^{\mathfrak{A}} \subseteq R_b$, then there is $b \in A$ such that $\bigcup_{x \in R} \varphi_x^{\mathfrak{A}} \subseteq R_b$.

The above conditions are the first order properties of R, since R is regular in $\mathfrak{A}$ if and only if the following sentences hold in $\mathfrak{A}$.

C'.1. $\forall x \exists y \neg R(y, x)$.

C'.2. $\forall x \forall y \exists z (R(x, z) \wedge R(y, z))$.

C'.3. $\forall v (\forall x \exists y (R(x, v) \Rightarrow \forall u (\varphi(x, u) \Rightarrow R(u, y))) \Rightarrow \exists y \forall x \forall u (R(x, v) \Rightarrow (\varphi(x, u) \Rightarrow R(u, y))))$.

Hence, if $\mathfrak{A} \equiv \mathfrak{B}$ (i.e. if $\mathfrak{A}$ is elementary equivalent to $\mathfrak{B}$), and R satisfies C.1., C.2., C.3., in $\mathfrak{A}$, then it does in $\mathfrak{B}$. In the next we assume that R is is regular in $\mathfrak{A}$.

L e m m a 1. 1° $\bigcup_{a \in A} R_a = A$.

2° *For every* $a_1, \ldots, a_n \in A$, $n \in \omega$, *there is* $b \in B$ *such that* $R_{a_1} \cup \cdots \cup R_{a_n} \subseteq R_b$, *and hence* $R_{a_1} \cup \cdots \cup R_{a_n} \neq A$.

P r o o f. 1° Immediately by C.2.

2° It is sufficient to prove that for any $a, b \in A$, there is $c \in A$ such that $R_a \cup R_b \subseteq R_c$, since the general case we can obtain easily by induction. Let $a, b \in A$. By C.2. there is $d \in A$ such that a, $b \in R_d$. Let $\varphi(x, y)$ be $R(y, x)$. Since for every $x \in R_c$, $\varphi_x^{\mathfrak{A}} = R_x$, by C.3. there is $c \in A$ such that $\bigcup_{x \in R_d} \varphi_x^{\mathfrak{A}} \subseteq R_c$. Since $a, b \in R_d$, we have $R_a \cup R_b \subseteq R_c$. $\dashv$

In order to $\mathfrak{A}$ has proper elementary R-end extension the theory $T = Th(\mathfrak{A}, a)_{a \in A} \cup \{\neg R(c, \boldsymbol{a}) \mid a \in A\}$, where c is a new constant symbol, should be consistent. As it is easily seen, T is consistent by lemma 1. and compactness theorem.

L e m m a 2. *Let* $\psi(x)$ *be a formula in one free variable in the language* $L_A = L \cup \{\boldsymbol{a} \mid a \in A\}$ *and* c *a new constant. Then* $\psi(c)$ *is inconsistent with* T *if and only if for some* $a \in A$ $\{x \in A \mid \mathfrak{A} \models \psi(x)\} \subseteq R_a$, i.e. $\mathfrak{A} \models \forall x (\psi(x) \Rightarrow R(x, \boldsymbol{a}))$

P r o o f. ($\Rightarrow$) Assume $\psi(c)$ is inconsistent with T. Then $T \models \neg \psi(c)$, hence for some $a_1, \ldots, a_n \in A$ $Th(\mathfrak{A}, a)_{a \in A} \models \neg R(c, \boldsymbol{a}_1) \wedge \cdots \wedge \neg R(c, \boldsymbol{a}_n) \Rightarrow \neg \psi(c)$. Therefore $\mathfrak{A} \models \forall y (\psi(y) \Rightarrow R(y, \boldsymbol{a}_1) \vee \cdots \vee R(y, \boldsymbol{a}_n))$. By Lemma 1 there is $b \in A$ such that $R_{a_1} \cup \cdots \cup R_{a_n} \subseteq R_b$. Then $\{y \in A \mid \mathfrak{A} \models \psi(y)\} \subseteq R_b$.

($\Leftarrow$) Assume $\psi(c)$ is consistent with T and for some $a \in A$ $\mathfrak{A} \models \forall x (\psi(x) \Rightarrow R(x, \boldsymbol{a}))$. Let $\mathfrak{B}$ be a model of $T \cup \{\psi(c)\}$, and let $c^{\mathfrak{B}} = c_0$. Then $\mathfrak{B} \models \psi(c_0)$. Since $\mathfrak{A} < \mathfrak{B}$ (i.e. $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$; we identify $\boldsymbol{a}^{\mathfrak{B}}$ with a for every $a \in A$), it follows that $\mathfrak{B} \models \forall y (\psi(y) \Rightarrow R(y, \boldsymbol{a}))$. Hence $\mathfrak{B} \models R(c_0, \boldsymbol{a})$ that contradicts to $\mathfrak{B} \models \neg R(c_0, a)$. $\dashv$

R e m a r k. We see that $\mathfrak{A} <_{eR} \mathfrak{B}$ if and only if there are no $a \in A$ and $b \in B$ such that $b \in R_a^{\mathfrak{B}}$ and $b \neq a'$ for all $a' \in R_a^{\mathfrak{A}}$. Hence $\mathfrak{A} <_{eR} \mathfrak{B}$ if and only if $\mathfrak{B}$ omits $\Sigma_a = \{R(x, \boldsymbol{a})\} \cup \{x \neq a' \mid \boldsymbol{a}' \in R_a\}$ for every $a \in A$.

L e m m a 3. *For every* $a \in A$ T *localy omits* Σ_a.

P r o o f. Let $\exists x \varphi(x, c)$ be consistent with T. Assume there is no $\sigma \in \Sigma_a$ so that $\exists x (\varphi(x, c) \wedge \neg \sigma)$ is consistent with T. Hence

1° $\exists x(\varphi(x, c) \wedge \neg R(x, a))$ is inconsistent with T, so by lemma 2, for some $b_1 \in A \{y \in A \mid \mathfrak{A} \models \exists x(\varphi(x, y) \wedge \neg R(x, a))\} \subseteq R_{b_1}$.

2° Let $a' \in R_a$. Then $\exists x(\varphi(x, c) \wedge x = a')$ is inconsistent with T i.e. $\varphi(a', c)$ is incosistent with T, so by lemma 2, there is $b \in A$ such that $\{y \in A \mid \mathfrak{A} \models \varphi(a', y)\} \subseteq R_b$. Therefore we have proved that for every $a' \in R_a$ there is $b \in A$ such that $\varphi_{a'}^{\mathfrak{A}} \subseteq R_b$, hence by regularity of R, there is $b_2 \in A$ such that $\bigcup_{a' \in R_a} \{y \in A \mid \mathfrak{A} \models \varphi(a', y)\} \subseteq R_{b_2}$.

Let $b \in A$ such that $R_{b_1} \cup R_{b_2} \subseteq R_b$. Then:

$$\{y \in A \mid \mathfrak{A} \models \exists x(\varphi(x, y) \wedge \neg R(x, a))\} \cup \{y \in A \mid \mathfrak{A} \models \exists x(\varphi(x, y) \wedge R(x, a))\} \subseteq R_b$$

i.e. $\{y \in A \mid \mathfrak{A} \models \exists x \varphi(x, y)\} \subseteq R_b$, so by lemma 2. $\exists x \varphi(x, c)$ is inconsistent with T, contradiction. $\dashv$

Theorem 2. *Let $\mathfrak{A}$ be a countable model and R a regular relation in $\mathfrak{A}$. Then $\mathfrak{A}$ has an elementary R-end extension.*

Proof. Immediately by remark above lemma 3, lemma 3 and theorem 1. $\dashv$

Corollary. *If R is regular in $\mathfrak{A}$, then there is $\mathfrak{B}$ such that $\mathfrak{A} <_{eR} \mathfrak{A}$ and* $|B| = \omega_1$.

Proof. In view of the theorem 2, we can construct an elementary chain $\mathfrak{A} = \mathfrak{A}_0 < \mathfrak{A}_1 < \cdots < \mathfrak{A}_\alpha < \cdots$, $\alpha < \omega_1$, so that $\mathfrak{A} <_{eR} \mathfrak{A}_{\alpha+1}$, $|A_\alpha| = \omega$, $A_\alpha \underset{\neq}{\subset} A_{\alpha+1}$. If $\lambda < \omega_1$ is a limit ordinal, then $\mathfrak{A}_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$ and it is easy to check that for $\alpha < \lambda$, $\mathfrak{A}_\alpha <_{eR} \mathfrak{A}_\lambda$. Then the required $\mathfrak{B} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$. $\dashv$

3. We are going to give some examples which illustrate the theorem 2.

1° Let $<$ be the natural strict ordering in Peano arithmetic. Obviously $<$ satisfies C.1, C.2. Condition 3 can be restated as $\forall v \psi(v)$, where $\psi(v)$ is

$$\forall x < v \exists y \forall u(\varphi(x, u) \Rightarrow u < y) \Rightarrow \exists y \forall x < v \forall u(\varphi(x, u) \Rightarrow u < y).$$

It is an easy exercise to prove by induction that $\forall v \psi(v)$ is a theorem in Peano arithmetic. Therefore $<$ is regular relation. Hence every countable model of Peano arithmetic allows elementary $<$-end extension. This statement is, of-course, a part of MacDowel and Specker's theorem, which says that every model of Peano arithmetic has an elementary $<$-end extension of the same cardinality ([1], p. 244, see also exercise 2.2.10 in [2]).

2° The relation of division $x \mid y \Leftrightarrow \exists z \; y = zx$ in natural numbers is regular (we take 1 as an initial element).

3° Let k be an inaccessible cardinal, X a set, $|X| = k$, and $S_k(X) = \{y \mid y \subseteq X, |y| < k\}$. Then $\subseteq$ is regular in $S_k(X)$. The regularity of $\subseteq$ follows from the following fact: Let $a \in S_k(X)$ and $\Phi = \{A_y \subseteq S_k(X) \mid y \subseteq a\}$ a family of sets. If for every $y \subseteq a$ A_y has an upper bound in $S_k(X)$, then $\bigcup_{y \subseteq a} A_y$ has an upper bound in $S_k(X)$.

4° Relation $\in$ in ZF, $ZF - P$ (ZF without power set axiom), $ZF - \infty$ (ZF without the axiom of infinity) is regular, as it can be easily seen. We see that C.3. is in fact the collection axiom. Therefore in each of these cases we can apply theorem 2 (see theorem 2.2.18 [2], also [3] for other similar construction).

5° Let k be a regular cardinal. Then the ordering $<$ of k is regular, since for $\alpha<k$, if for every $\xi<\alpha$ $S_\xi \subseteq k$, has an upper bound in k, then $\bigcup_{\xi<\alpha} S_\xi$ has an upper bound in k itself. At this point we can derive Keisler's two cardinal theorem (theorem 3.2.14, [2], p. 135) which says: Let $\mathfrak{A}=(A, V, \ldots)$ be a model in a countable L such that $\omega \leqslant |V|<|A|$. Then there are two models $\mathfrak{B}=(B, W, \ldots)$ and $\mathfrak{C}=(C, W, \ldots)$ such that $\mathfrak{B}<\mathfrak{A}$, $|B|=\omega$, $\mathfrak{B} \prec \mathfrak{C}$ and $|C|=\omega_1$. Proof of Keisler's theorem: By downward Löwenheim-Skolem-Tarski theorem we may assume that $|A|=|V|^+$. Let us consider the expansion $(\mathfrak{A}, <)$, where $<$ is the ordering of the cardinal $|A|$ and let $(\mathfrak{B}, <^{\mathfrak{B}}) \prec (\mathfrak{A}, <)$, where $|B|=\omega$. We have remarked already that $<^{\mathfrak{B}}$ is still regular. Since $V \subseteq A$, $|V|<|A|$ and $|A|$ is a regular cardinal there is $a \in A$ such that $V \subseteq \{x \in A \mid x<a\}$. Hence there is $b \in B$ such that $W \subseteq \{x \in B \mid x<^{\mathfrak{B}} b\}$ $(W=V^{\mathfrak{B}})$. Since $<^{\mathfrak{B}}$ is a regular relation, it follows by corollary that there is model $(\mathfrak{C}, <^{\mathfrak{C}})$ which is an elementary $<$-end extension of $(\mathfrak{B}<^{\mathfrak{B}})$ such that $|C|=\omega_1$. Since $W \subseteq \{x \in B \mid x<^{\mathfrak{B}} b\}$, it follows that $V^{\mathfrak{C}} \subseteq \{x \in B \mid x<^{\mathfrak{B}} b\}$, and hence $V^{\mathfrak{C}}=W$. $\dashv$

6° Let k be a regular cardinal and L_k $k-th$ constructible set in the constructible hierarchy. Since $L_k \models ZFL-P$, it follows that $\in$ is regular in L_k. Similarly, if $H_k=\{x \mid \text{transitive closure of } x \mid <k\}$, then $(H_k, \in) \models ZF-P$ so $\in$ is regular in H_k.

One interesting question related to the last example may be asked: If $(M, \in)$ is a countable transitive model of ZF, or $ZF-P$, does there exist a *transitive* elementary end extension $(N, \in)$ of $(M, \in)$? In the view of Mostowski's collapsing lemma it is sufficient $(M, \in)$ to have a well founded elementary end extension.

REFERENCES

[1] J. L. Bell, A. B. Slomson, *Models and Ultraproducts* (North-Holland, Amsterdam) (1971).

[2] C, C, Chang, H. J. Keisler, *Model Theory* (North-Holland, Amsterdam) (1973)

[3] H. J. Keisler and M. Morley, *Elementary extensions of models of set theory*, Israel J. Math. 6, 49—65 (1968)

Ovaj rad finansirala je Republička zajednica za naučni rad SR Srbije