PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 21 (35), 1977, pp. 135—140

SOME REMARKS ON BOOLEAN TERMS — MODEL THEORETIC APPROACH

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(Communicated March 2, 1976)

In this article we discuss mainly some properties of Boolean terms from model theoretic point of view. In this way, among others, we prove representation theorem for Boolean terms and also Boole-Schröder consistency condition for Boolean equations. In part 3. we prove that a Boolean map is $1-1$ if and only if it is onto. It is also shown that this consideration can be extended on some other structures. In part 4. it is shown that the countable free Boolean algebra with induced ordering is an universal model for partial orderings. The main tool that we use are theorems T. 1. 1—4.

1. Once S. Prešić applied Horn formulas in order to describe solutions of Boolean equations. Motivated by the above idea we demonstrate that kind of model theoretic approach in exhibiting some properties of mappings determinated by Boolean terms, or in some cases by terms of an arbitary language L. In spite of the fact that the most of cited statements are already known, there could be some interest in the presented method of their proofs.

Before we proceed further, we list some theorems that we shall use later.

1.1. Theorem (Horn). *Horn sentences are preserved under direct products.* $\dashv$

1.2. Theorem (Vaught). *If* θ *is a Horn sentence of the language of Boolean algebras and* $\mathbf{2}\models\theta$, *then* θ *holds on all Boolean algebras.* $\dashv$
$\mathbf{2}$ *is here two-element Boolean algebra.*

1.3. Theorem. *If* $\boldsymbol{B}$ *is a Boolean algebra and* $\boldsymbol{B}'\subseteq\boldsymbol{B}$ *finitelly generated subalgebra, then* $\boldsymbol{B}'$ *is finite.* $\dashv$
Hence, $\boldsymbol{B}$ *is a direct limit of finite Boolean algebras.*

1.4. Theorem. *Assume that* $\Phi=(\mathfrak{A}_i, h_{ij}: i\leqslant j\in I)$ *is a direct system so that each* h_{ij} *is a monomorphis and* $\mathfrak{A}_\infty=\underset{\rightarrow}{\lim}\,\Phi$. *If* θ *is a* Π_2 (i. e. universal-existential) *sentence that holds on all models* $\mathfrak{A}_i$ (or, if there is a coffinal subset $I'\subseteq I$ so that θ holds on all $\mathfrak{A}_i$ for $i\in I'$) *then* $\mathfrak{A}_\infty\models\theta$. $\dashv$.

We shall use the following symbols.

Boolean multiplication is denoted by ., addition by $+$ and complement of x by x'. Infimum of the set $X \subseteq B$ is denoted by $\prod_{x \in X} x$ and supremum by $\sum_{x \in X} x$.

By $\bar{x}$, $\bar{y}$ etc., we denote finite sequences $x_1, \ldots, x_n$, $y_1, \ldots, y_m$ if there is no ambiguity in use of m, n. $\alpha \in 2^n$ stands for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in 2$. In this case, by definition $\bar{x}^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$, where $x^\circ = x'$, $x^1 = x$. Also for $\alpha, \beta \in 2^n$ $\beta^\alpha = \beta_1^{\alpha_1} \cdot \ldots \cdot \beta_n^{\alpha_n}$.

Meta-equality is denoted by $\doteq$.

2. Assume that $\boldsymbol{B}$ is a Boolean algebra and $t(\bar{x})$ a Boolean term over $\boldsymbol{B}$, that is t is a term of the language $\{+, \cdot, '\} \cup \{\underline{a} : a \in B\}$. It will be shown that some properties of the term t are in fact transferred from Boolean algebra **2**. For example, in this way is obtained Rudeanu's representation theorem for Boolean terms.

Let $t(\bar{x}, \bar{y})$ be a Boolean term and

$$\varphi_t \doteq \forall \bar{x} \forall \bar{y} \left(\bigwedge_{\alpha \in 2^n} t(\alpha, \bar{y}) = 0 \Rightarrow t(\bar{x}, \bar{y}) = 0 \right).$$

It is easy to check that the following holds:

2.1. $\mathbf{2} \models \varphi_t$.

2.2. The sentence φ_t is a Horn formula. Therefore, according to T. 1. 2. it holds on every Boolean algebra.

Hence, we have the following assertion:

2.3. Proposition. *Let $t(\bar{x}, \bar{a})$ be a Boolean term over a Boolean algebra $\boldsymbol{B}$, $\bar{a} \in \boldsymbol{B}$. Then the following holds:*

If for all $\alpha \in 2^n$ $\boldsymbol{B} \models t(\alpha, \underline{\bar{a}}) = 0$ then $\boldsymbol{B} \models \forall \bar{x}\, t(\bar{x}, \underline{\bar{a}}) = 0$. $\dashv$

2.3.1 Corollary. *Let $t_1(\bar{x})$, $t_2(\bar{x})$ be Boolean terms over $\boldsymbol{B}$. If for all $\alpha \in 2^n$ $\boldsymbol{B} \models t_1(\alpha) = t_2(\alpha)$ then $\boldsymbol{B} \models \forall \bar{x}\, t_1(\bar{x}) = t_2(\bar{x})$.*

Proof. We just apply the above proposition on $t_1 \Delta t_2$, where Δ is symmetric difference. $\dashv$

2.3.2 Corollary (S. Rudeanu). *Let $t(\bar{x})$ be a Boolean term over a Boolean algebra $\boldsymbol{B}$ and $\tilde{t}(\bar{x}) = \sum_{\alpha \in 2^n} t(\alpha) \cdot \bar{x}^\alpha$. Then*

$\boldsymbol{B} \models \forall \bar{x}\, t(\bar{x}) = \tilde{t}(\bar{x})$.

Proof. Let $\beta \in 2^n$. Then

$\tilde{t}(\beta) = \sum_{\alpha \in 2^n} t(\alpha) \cdot \beta^\alpha = t(\beta)$, hence

$\boldsymbol{B} \models \bigwedge_{\beta \in 2^n} t(\beta) = \tilde{t}(\beta)$, so by corollary 2.3.1

$\boldsymbol{B} \models \forall \bar{x}\, t(\bar{x}) = \tilde{t}(\bar{x})$. $\dashv$

2.3.3 Corollary. *Let $t(\bar{x})$ be a Boolean term over $\boldsymbol{B}$. Then there is a Boolean term $h(\bar{x})$ over $\boldsymbol{B}$ so that*

$$\boldsymbol{B} \models \forall \bar{x}\, h(\bar{x}) \cdot t(\bar{x}) = \prod_{\alpha \in 2^n} t(\alpha).$$

Proof. Let $g(\bar{x})$ be defined for $\beta \in 2^n$ by $g(\beta) = \prod_{\alpha \in 2^n - \{\beta\}} t(\alpha)$ and $h(\bar{x}) = \sum_{\beta \in 2^n} g(\beta) \cdot \bar{x}^{\beta}$. Therefore, for each $\beta \in 2^n$

$$\boldsymbol{B} \models h(\beta) \cdot t(\beta) = \prod_{\alpha \in 2^n} t(\alpha).$$ Hence by corollary 2.3.2

$$\boldsymbol{B} \models \forall \bar{x}\, h(\bar{x}) \cdot t(\bar{x}) = \prod_{\alpha \in 2^n} t(\alpha). \dashv$$

2.4 Proposition (Boole, Schröder). *Let $t(\bar{x})$ be a Boolean term over $\boldsymbol{B}$. Then the equation $t(\bar{x}) = 0$ has solution in $\boldsymbol{B}$ if and only if $\prod_{\alpha \in 2^n} t(\alpha) = 0$. In other words, $\boldsymbol{B} \models \exists x t(\bar{x}) = 0 \Leftrightarrow \prod_{\alpha \in 2^n} t(\alpha) = 0$.*

Proof. 1° The part $\boldsymbol{B} \models \exists \bar{x}\, t(\bar{x}) = 0 \Rightarrow \prod_{\alpha \in 2^n} t(\alpha) = 0$ follows from the corollary 2.3.3.

2° Let us prove $\boldsymbol{B} \models \prod_{\alpha \in 2^n} t(\alpha) = 0 \Rightarrow \exists \bar{x}\, t(\bar{x}) = 0$.

Consider the following Horn sentence

$$\psi \doteq \forall u_1 u_2 \cdots u_{2n} \exists x_1 x_2 \cdots x_n \left(\sum_{\alpha \in 2^n} u_\alpha = 1 \Rightarrow \bigwedge_{\alpha \in 2^n} (\bar{x}^\alpha \leqslant u_\alpha) \right).$$

Since for $\alpha, \beta \in 2^n$ $\beta^\alpha = 1$ in case $\alpha = \beta$, $\beta^\alpha = 0$ otherwise, it follows $2 \models \psi$. Therefore, by theorem 1.2 ψ holds on all Boolean algebras.

Assume that $\boldsymbol{B} \models \prod_{\alpha \in 2^n} t(\alpha) = 0$. Therefore $\boldsymbol{B} \models \sum_{\alpha \in 2^n} t'(\alpha) = 1$.

Since ψ holds on $\boldsymbol{B}$, there is $\bar{c} \in \boldsymbol{B}$ so that $\bar{c}^\alpha \leqslant t'(\alpha)$ for all $\alpha \in 2^n$. Hence for each $\alpha \in 2^n$ $t(\alpha) \cdot \bar{c}^\alpha = 0$. Thus $\sum_{\alpha \in 2^n} t(\alpha) \cdot \bar{c}^\alpha = 0$, so by corollary 2.3.2 $t(\bar{c}) = 0$. $\dashv$

According to the last proposition, the equation $t(\bar{x}) = 0$ has no solution in B if and only if there is $c \in B - \{0\}$ and a Boolean term h over B such that $B \models \forall \bar{x}\, h(x) \cdot t(\bar{x}) = c$.

It should be remarked that all above statements can be proved using only theorems 1.1,3,4 since all the formulas in question are in fact Horn universal-existential sentences.

2.5 Example. Boolean equation $f(x) = 0$ has solution if and only if $f(0) \cdot f(1) = 0$. If $f(0) \cdot f(1) = 0$, then the general solution of above equation is $x = u \Delta f(u)$, $u \in B$, since in this case, as it is easily seen, $x = x \Delta f(x)$ is a reproductive equation which is equivalent to $f(x) = 0$.

2.6 Remark. A function $f: A \to A$ is reproductive if $f^2 = f$.

2.7. Theorem (S. Prešić). *If f is a reproductive function, then the general solution in A of $x=f(x)$ is given by $x=f(u)$, $u\in A$.*

2.8. Remark. By repeated use of consistency condition given by proposition 2.4 and above example, one can solve any Boolean equation $t(\bar{x})=0$ over an arbitrary Boolean algebra B.

3. In [2a] it is shown that if $f: C^n \to C^n$ (C is the field of complex numbers) is a polynomial map then: If f is $1-1$ then f is onto. In this part we are giving more examples of similar nature.

Assume that $\mathfrak{A}$ is a model for a language L. A map $f: A^n \to A^n$ is an L-map over $\mathfrak{A}$ if there are terms $t_1, \ldots, t_n$ of the language $L_A = L\cup\{\underline{a}: a\in A\}$ so that for all $\bar{a}\in A$

$$f(\bar{a})=(t_1(\bar{a}), \ldots, t_n(\bar{a})).$$

3.1 Proposition. *Assume that a model $\mathfrak{A}$ is a direct limit of finite models. Then for every L-map f over $\mathfrak{A}$ the following holds:* (*) *If f is $1-1$ then f is onto.*

Proof. Let $f(\bar{x})=(t_1(\bar{x}, \bar{a}), \ldots, t_n(\bar{x}, \bar{a}))$ where $\bar{a}\in A$ and $t_i(\bar{x}, \bar{y})$ are terms of L. Consider the following sentence

$$\varphi \doteq \forall \bar{y}\,(\forall \bar{x}\,\forall \bar{z}\,(\bigwedge_{i=1}^{n} t_i(\bar{x}, \bar{y})=t_i(\bar{z}, \bar{y}) \Rightarrow x_1=z_1 \wedge \cdots \wedge x_n=z_n) \Rightarrow \forall \bar{u}\,\exists \bar{v} \bigwedge_{i=1}^{n} u_i=t_i(\bar{v}, \bar{y}))$$

Let us observe the following facts:

1° φ is Π_2 sentence,

2° φ holds on all finite models for L.

3° $\mathfrak{A}\models\varphi$ if and only if for every $\bar{b}\in A$ the map $g(\bar{x})=(t_1(\bar{x}, \bar{b}), \ldots, t_n(\bar{x}, \bar{b}))$ if it is $1-1$ then it is onto.

Therefore by theorem 1.4 φ holds on $\mathfrak{A}$, hence (*) holds. ⊣

3.1.1 Corollary. *Let $f: B^n \to B^n$ be a Boolean map over a Boolean algebra B. By theorems* 1.3, 1.4 *and the last proposition follows: If f is $1-1$ then f is onto.*

The above proposition can be applied also in these cases:

1° If $(G, +, 0)$ is an Abelian group in which all elements are of order p (p is a prime) and

$$f(x)=(\underline{m}_1^1 x_1+\cdots+\underline{m}_n^1 x_n+a_1, \ldots, \underline{m}_1^n x_1+\cdots+\underline{m}_n^n x_n+a_n),$$

m_j^i are integers and $\bar{a}\in G$.

2° If $(G, +, 0)$ is an Abelian group so that every element of G is of finite order.

3° Let G_p be a cyclic group of order p (p is a prime) and $G=\prod_{p\in P} G_p/F$ where P is the set of primes and F a nonprincipal ultrafilter over P.

Then G is a divisible torsion-free Abelian group and for every $\{+\}$-map f over G (*) holds. Since the theory of divisible Abelian groups is complete, (*) holds for any $\{+\}$-map over arbitrary Abelian divisible group.

4° By means of Poor Man's Lefschetz Principle see [2*a*, *b*], (*) holds for any polynomial map over any algebraically closed field (see [2 *a*]).

It could be of some interest to consider the converse of (*), namely

(**) If f is onto then f is $1-1$.

The notion of (**) is formalized by the sentence

$$\psi \doteq \forall \bar{y} (\forall \bar{u} \exists \bar{v} \bigwedge_{i=1}^{n} u_i = t_i(\bar{v}, \bar{y}) \Rightarrow \forall \bar{x} \forall \bar{z} (\bigwedge_{i=1}^{n} t_i(\bar{x}, \bar{y}) = t_i(\bar{z}\, \bar{y}) \Rightarrow x_1 = z_1 \wedge \cdots \wedge x_n = z_n)).$$

By inspection it is easily seen that ψ is a Horn sentence. Hence ψ is preserved under (reduced) products of models.

3.2. Proposition. *Assume that B is a Boolean algebra and $f\colon B^n \to B^n$ a Boolean map. Then: If f is onto then f is $1-1$.*

Proof. Obviously $2 \models \psi$. By theorem 1.2 ψ holds on all Boolean algebras and therefore (**) holds. ⊣

3.2.1 Corollary (Whitehead, Löwenheim). *Let $f\colon B^n \to B^n$ be a Boolean map. Then: f is $1-1$ if and only if it is onto.* ⊣

3.3 Remark. For all historical and other facts that concern Boolean functions, maps and equations one should consult [5]. Similar propositions can be stated for examples 1°, 3°. For example, let G be an Abelian group so that all elements of G are of order p. Since there are infinitelly many nonisomorphic such finite groups, by compactness theorem ψ holds at least on one infinite Abelian group with all elements of order p. The first order theory of these groups is k-categorical for $k \geqslant \omega$, hence ψ holds on all infinite Abelian groups with all elements of order p, so (**) holds.

With similar argument it can be shown that (**) also holds in the case of divisible torsion-free Abelian groups.

4. Once Đ. Kurepa has asked if there is a countable universal model for partial orderings. We show that countable atomless Boolean algebra Ω with induced ordering is such a one. On the first sight the consideration that follows seems to be not connected with previous parts of this article, but we remark that every countable atomless Boolean algebra is free, so it is in fact the algebra of Boolean terms (over a countable set of variables).

Using compactness theorem it is easily shown the following

4.1 Proposition. *Assume that $\mathfrak{A}$ is a direct limit (of a direct system with monomorphisms) of submodels of a theory T. Then $\mathfrak{A}$ is a submodel of T.* ⊣

Above proposition is a slight modification of Tarski-Maljcev theorem: *If every finitely generated submodel of* $\mathfrak{A}$ *is a submodel of* T, *then* $\mathfrak{A}$ *is a submodel of* T.

4.2 Proposition. *Let* Ω *be a countable free Boolean algebra and* $(\Omega, \leqslant)$ *induced ordering. If* $\mathfrak{A}=(A, \leqslant)$ *is a countable partial ordering then* $\mathfrak{A}$ *is embedded into* $(\Omega, \leqslant)$.

Proof. Let us observe that every finitely generated submodel of $\mathfrak{A}$ is finite, hence $\mathfrak{A}$ is a direct limit of finite partial orderings. Every finite partal ordering $(X, \leqslant)$ is embedded into some finite Boolean algebra (that embedding is realized by $f(x)=\{y \in X: y \leqslant x\}$, $f:(X, \leqslant) \rightarrow (S(X), \subseteq)$, where $S(X)$ is the partitive set of X) and therefore into $(\Omega, \leqslant)$. By proposition 4.1 $\mathfrak{A}$ is embedded into some model elementary equivalent to $(\Omega, \leqslant)$, and by downward Löwenheim-Skolem-Tarski theorem into countable one. Since the theory of atomless Boolean algebras is ω-categorical, it follows that $\mathfrak{A}$ is embedded into countable free Boolean algebra. $\dashv$

4.3 Remark. If $\mathfrak{A}$ is a distributive lattice it can be shown that the embedding in question can be chosen so that it preserves finite infimum and supremum.

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