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REDUCED PRODUCTS OF SATURATED INTUITIONISTIC THEORIES

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In [1] Aczél proposed a use of saturated (intuitionistic) theories instead of Kripke-models as a (classical) semantics for intuitionistic logic. He also suggested that a construction analogous to that of ultraproduct could be carried out by means of saturated theories. In the present paper such a construction is proved to be valid and as an example of its applicability a proof of compactness theorem for first-order intuitionistic logic is given.*)

We shall first give a short account of some notions defined in [1] and state (without proof) the basic theorems from [1]. If A is a sentence (closed first-order formula), $\operatorname{Ind} A$ denotes the set of all individual constants occuring in A. If Γ is a set of sentences, $\operatorname{Ind}(\Gamma)=\bigcup\{\operatorname{Ind}(A):A\in\Gamma\}$ and $Cn(\Gamma)=\{A:\operatorname{Ind}(A)\in\operatorname{Ind}(\Gamma)$ and $\Gamma\vdash A\}$ ($\vdash$ denotes intuitionistic consequence relation). A set of sentences Γ is called *saturated theory* iff it is *deductively closed* ($Cn(\Gamma)=\Gamma$), *consistent* ($\Lambda\notin\Gamma$, where Λ denotes absurdity), *prime* (if $A\vee B\in\Gamma$ then $A\in\Gamma$ or $B\in\Gamma$) and *existential* (if $\exists xA(x)\in\Gamma$ then $A(a)\in\Gamma$ for some $a\in\operatorname{Ind}(\Gamma)$). If Γ is a consistent set of sentences let $S_\Gamma=\{\Delta:\Gamma\subseteq\Delta$ and Δ is saturated$\}$ and let $\Gamma\models A$ mean $A\in\Delta$ for all $\Delta\in S_\Gamma$, such that $\operatorname{Ind}(A)\subseteq\operatorname{Ind}(\Delta)$.

Lemma 1. (*Strong semantic completeness*) (Aczél, Fitting, Thomason) *$\Gamma\vdash A$ iff $A\in\Delta$, for all $\Delta\in S_\Gamma$ such that* $\operatorname{Ind}(A)\subseteq\operatorname{Ind}(\Delta)$.

Let S be the class of all saturated Γ with $\operatorname{Ind}(\Gamma)\neq\varnothing$. For $\Gamma\in S$ let $\mathcal{A}_\Gamma=(\operatorname{Ind}(\Gamma), \{P_\Gamma:P\in\operatorname{Rel}\})$ be a classical relational structure, where Rel *is the set of all relational symbols and $P_\Gamma=\{\langle a_1\ldots a_n\rangle:P(a_1\ldots a_n)\in\Gamma\}$. Then $\mathcal{S}=(\{\mathcal{A}_\Gamma:\Gamma\in S\},\subseteq)$ is a Kripke structure, with forcing relation defined for atomic sentences A (with $\operatorname{Ind}(A)\subseteq\operatorname{Ind}(\Gamma)$) by $\mathcal{A}_\Gamma\Vdash A$ iff $A\in\Gamma$. Let $\operatorname{Val}(\mathcal{S},\Gamma)=\{A:\mathcal{A}_\Gamma\Vdash A\}$.*

Lemma 2. (Aczél) *Γ is saturated iff* $\operatorname{Val}\mathcal{S},\Gamma)=\Gamma$.

Lemma 3. *For every Kripke model (K, i), (i being a node in the tree K),* $\operatorname{Val}(K, i)$ *is a saturated theory.*

Let $\{\Delta_i:i\in I\}$ be a collection of saturated theories and let $L_i=\operatorname{Ind}(\Delta_i)$. We assume $L_i\neq\varnothing$ for all $i\in I$. $\prod_{i\in I} L_i$ be the direct product of sets L_i defined

*) I wish to thank Dr. Aleksandar Kron who suggested to me the use of prime filters instead of ultrafilters.

in the usual way and let F be a prime filter on I. We define usual equivalence relation on $\prod L_i : a \sim b$ iff $\{i : a^i = b^i\} \in F$ (by a^i the i-th coordinate of $a\, (\in \prod L_i)$ is understood), and corresponding equivalence classes $a_{/\sim} = \{b \in \prod L_i : a \sim b\}$.

Now we define a reduced product of sets L_i over the prime filter $F : L = \prod L_{i/F} = \{a_{/\sim} : a \in \prod L_i\}$. Let $\Phi = \{A : \mathrm{Ind}\,(A) \subseteq L\}$, and for $A\, (a_{1/\sim}, \ldots, a_{n/\sim}) \in \Phi$ let $A^i = A\,(a_1^i, \ldots, a_n^i)$. Finally we define:

$$\prod_{i \in I} \Delta_{i/F} = \{A \in \Phi : \{i : A^i \in \Delta_i\} \in F\}$$

Theorem 1. *$\prod_{i \in I} \Delta_{i/F} = \Delta$ is a saturated theory.*

Proof. 1) Δ is a theory, i.e. $Cn\,(\Delta) = \Delta$. Obviously $\Delta \subseteq Cn\,(\Delta)$. Suppose $A \in Cn\,(\Delta)$. This means that there are $B_1, \ldots B_n$ such that $\vdash (B_1 \& \ldots \& B_n) \to A$. According to Lemma 1. $\vdash C$ iff $C \in \Gamma$ for every saturated theory Γ such that $\mathrm{Ind}\,(C) \subseteq \mathrm{Ind}\,(\Gamma)$, so we have $(B_1^i \& \ldots \& B_n^i) \to A^i \in \Delta_i$ for all $i \in I$. On the other hand $B_k \in \Delta\, (k \in \{1, \ldots, n\})$ iff $f_k = \{i : B_k^i \in \Delta_i\} \in F$. So $\{i : B_1 \& \ldots \& B_n^i \in \Delta_i\} = f_1 \cap \cdots \cap f_n$, as Δ_i are theories, and since F is a filter $f = f_1 \cap \cdots \cap f_n = F$. So we have

$$\{i : A^i \in \Delta_i\} \supseteq \{i : B_1^i \& \ldots \& B_n^i \in \Delta_i\} \cap \{i : (B_1^i \& \ldots \& B_n^i) \to A^i \in \Delta_i\} = f \cap I = f \in F$$

and that means $A \in \Delta$.

2) Δ is consistent. Since all Δ_i are consistent, i.e. $\Lambda \notin \Delta_i$ for all $i \in I$, it follows from the definition of Δ that $\Lambda \notin \Delta$.

3) Δ is prime. Let $A \vee B \in \Delta$. That means $\{i : A^i \vee B^i \in \Delta_i\} \in F$. Since all Δ_i are prime theories, $A^i \vee B^i \in \Delta_i$ iff $A^i \in \Delta_i$ or $B^i \in \Delta_i$. If $f_1 = \{i : A^i \in \Delta_i\}$ and $f_2 = \{i : B^i \in \Delta_i\}$ then obviously $f_1 \cup f_2 = \{i : A^i \vee B^i \in \Delta_i\}$ $(\in F)$. As F is supposed to be a prime filter, it follows that $f_1 \in F$ or $f_2 \in F$, i.e. $A \in \Delta$ or $B \in \Delta$.

4) Δ is existential. In proving this fact we make use of a version of Axiom of Choice. Let $(\exists x)\, A\,(x) \in \Delta$. We shall show the existence of an $a_{/\sim} \in L$, such that $A\,(a_{/\sim}) \in \Delta$. $(\exists x)\, A\,(x) \in \Delta$ by definition means $\{i : (\exists x)\, A^i\,(x) \in \Delta_i\} \in F$. Since every saturated theory is existential, $(\exists x)\, A^i\,(x) \in \Delta_i$ implies $A^i\,(c_i) \in \Delta_i$ for some $c_i \in L_i$. By the Axiom of Choice, there exists an $a \in \prod_{i \in I} L_i = L$, so that $a_j = c_j$ for all $j \in \{i : (\exists x)\, A^i\,(x) \in \Delta_i\}$. Then, obviously, $\{i : A^i\,(a_i) \in \Delta_i\} \in F$, i.e. $A\,(a_{/\sim}) \in \Delta$.

Definition. We say that the saturated theory $\Delta = \prod_{i \in I} \Delta_{i/F}$ is the reduced product of the collection of saturated theories $\{\Delta_i : i \in I\}$ over the prime filter F or shorter *the prime product*.

Compactness theorem. In this section we shall consider only sentences without individual constants, i.e. $A^i = A$ and $A \in \Delta$ iff $\{i : A \in \Delta_i\} \in F$.

We say that a sentence A is satisfiable iff there is a Kripke model $(\boldsymbol{K}, \boldsymbol{i})$ such that $\boldsymbol{i} \Vdash A$. A set of sentences Σ is satisfiable iff there is a Kripke model $(\boldsymbol{K}, \boldsymbol{i})$ such that $\Sigma \subseteq \mathrm{Val}\,(\boldsymbol{K}, \boldsymbol{i}) = \{A : i \Vdash A\}$.

T h e o r e m (Compactness) *A set of sentences* Σ (without individual constants) *is satisfiable iff every finite subset of* Σ *is satisfiable.*

P r o o f. One direction is obvious. Let $I=\{\Delta:\Delta\subseteq\Sigma$ and Δ is finite$\}$. By hypothesis for every $\Delta\in I$ there is a Kripke model $(K,i)_\Delta$ such that $\Delta\subseteq\subseteq\mathrm{Val}(K,i)_\Delta=\Gamma_\Delta$. By Lemma 3. Γ_Δ is saturated. For $\Delta\in I$ let $\Delta^*=\{\Delta'\in I::\Delta\subseteq\Delta'\}$. The set $\{\Delta^*:\Delta\in I\}$ has a finite intersection property, so that it can be a enlarged to a prime filter F. Let $L_\Delta=\mathrm{Ind}(\Gamma_\Delta)$ and $L=\prod_{\Delta\in I}L_{\Delta/F}$. Then the prime product $\Gamma=\prod_{\Delta\in I}\Gamma_{\Delta/F}$ is a model of Σ, i.e. $\Sigma\subseteq\Gamma$ and $(\mathcal{S},\Gamma)$ is a Kripke model for Σ (since for any $A\in\Sigma$, $\Delta\in I: A^\Delta\in\Gamma_\Delta\}=\{\Delta:A\in\Gamma_\Delta\}\supseteq\{A\}^*\in F)$.

REFERENCES

[1] P. H. G. A c z é l, *Saturated intuitionistic theories, in Contributions to Mathematical Logic*, ed. H. A. Shmidt, K. Shutte and H. J. Thiele, North Holland, Amsterdam, 1968, pp. 1—11.

[2] J. L. B e l l and A. B. S l o m s o n, *Models and Ultraproducts*, North Holland, Amsterdam, 1971.

[3] M. F i t t i n g, *Intuitionistic Logic Model Theory and Forcing*, North-Holland, Amsterdam, 1969.

[4] S. K r i p k e, *Semantical analysis intuitionistic logic I, in Formal Systems and Recursive Functions*, ed. J. N. Crossley and M. A. E. Dummet, North Holland, Amsterdam, 1965, pp. 92—130.