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REGULAR VARIATION AND ASYMPTOTIC PROPERTIES OF SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. This paper is concerned with asymptotic properties of positive solutions tending to zero, for $x \to \infty$, of the equation

$$y'' = f(x)\,\varphi(y); \tag{1.1}$$

$f(t)$ and $\varphi(t)$ are continuous and postive for $t>0$.

The existence of such solutions is guaranteed e. g. by the following result of Wong, [1]: Let $t^{-1}\varphi(t)$ increase, then the equation (1.1) has solutions tending to zero if and only if $\int^{\infty} tf(t)\,dt$ diverges.

We assume that f and φ belong to a class of functions of frequent use in various branches of analysis and of stochastic processes, generally called regularly varying functions (in the sense of Karamata who introduced them in 1930, [2]). The recent treatise of Seneta [3] covers the basic theory of such functions, and we present here some of the definitions and properties which are needed to formulate and prove our results:

Definition 1. *A positive continuous function* ρ *defined on* (a, ∞) $a \geqslant 0$ *is said to be o-regularly varying at infinity if for all* $\lambda > 0$

$$\lim_{x\to\infty} \frac{\rho(\lambda x)}{\rho(x)} = h(\lambda)$$

where $0 < h(\lambda) < \infty$.

It is known that $h(\lambda) = \lambda^{\sigma}$ and σ is called the index of regular variation of ρ.

All rational or, more generally, explicit algebraic functions are such, this is also true for the functions

$x^s(2+\sin x)$, $\int^{x} (\ln x)^{-1}\, dx$, etc.

Definition 2. *A positive continuous function L defined on* (a, ∞) *is said to be slowly varying at infinity if for all* $\lambda > 0$

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1.$$

All positive functions tending to positive constants are such, this is also true for powers of iterated logarithms or for the function $(2+x^{-1}\sin x)\log(x+2)+\cos x$ etc.

It follows that $\rho(x) = x^{\sigma} L(x)$ and that a slowly varying function is an o-regularly varying one of index $\sigma = 0$.

One of the basic properties of such functions is the following:

For $\sigma > -1$

$$\int_a^x t^{\sigma} L(t)\, dt \sim \frac{1}{\sigma+1} x^{\sigma+1} L(x), \quad x\to\infty, \tag{1.2}$$

for $\sigma = -1$

$$\int_a^x \frac{L(t)}{t}\, dt = L_1(x) \to \infty, \quad x\to\infty \tag{1.3}$$

where $L_1(t)$ is a new slowly varying function such that $L_1(t)/L(t) \to \infty$, $x\to\infty$, whose behavior cannot be expressed by a single formula for any $L(t)$, which is shown e.g. by the examples $L(t) = \ln t$, and $L(t) = (\ln t)^{-1}$ when for $x\to\infty$

$$\int_a^x t^{-1} L(t)\, dt \sim \frac{1}{2} \ln^2 x \text{ and } \int_a^x t^{-1} L(t)\, dt \sim \ln x$$

respectively.

The following result, more general then (1.2), holds, [4]:

Proposition: 1. *Let* $\int_a^{\infty} t^{\eta} |f(t)|\, dt$ *converge for some* $\eta > 0$, *then*

$$\int_a^{\infty} f(t) L(\lambda t)\, dt \sim L(\lambda) \int_a^{\infty} f(t)\, dt, \ \lambda\to\infty.$$

We need also [3 p. 52]:

Proposition 2. *For any* $\varepsilon > 0$, $x^{-\varepsilon} L(x)$ *is almost decreasing and* $x^{\varepsilon} L(x)$ *is almost increasing for* $x \geqslant x_0$.

In that we say (following S. Bernstein) that g is almost increasing if $x_1 < x_2$ implies $g(x_1) < A g(x_2)$, $A > 1$; almost decreasing functions are defined likewise.

By replacing the limiting procedure in Definition 1 by a boundedness condition (holding for a finite interval of λ) one generalizes the preceding class as follows [5], [6] (Cf. [3]):

Definition 3. *A positive continuous function g defined on* (a, ∞) *is said to be O-regularly varying at infinity, if*

$$m \leq g(\lambda x)/g(x) \leq M, \qquad 1 \leq \lambda \leq a$$

where m, M and a are any constant such that $0<m<1$, $1<M<\infty$, $1<a<\infty$.

All positive functions bounded away from both 0 and ∞ are such, this is also true for the functions $2+\sin x$, $x^s\{1+p\sin(2\pi \log x)\}$ with small p etc.

Another characterization of such functions is more suitable for our purposes [7], [8]:

Proposition 3. *A positive continuous function g defined on* (a, ∞) is 0-*regularly varying at infinity if and only if there exist real numbers* p, q $q<p$ *such that* $x^p g(x)$ *is almost increasing and* $x^q g(x)$ *is almost decreasing for* $x \geq x_0$.

The proof of the above result can be found in [7] together with some additional properties of functions in question. We conclude by the remark that regular variation of $g(x)$ (both $o-$ and $O-$) at 0 is defined as the one at infinity of $g\left(\frac{1}{x}\right)$.

2. Results. These consist of asymptotic estimate of $y(x)$ for large x, expressed by inequalities, and of precise asymptotic behavior, but it turns out that the former play a fundamental role in the whole consideration.

2.1 *Asymptotic estimate.* The core of the subject is the

Theorem 1. *Let* $f(x)$ *be O-regularly varying at infinity and such that*

$$\int_a^{\infty} tf(t)\,dt = \infty, \tag{2.1}$$

and let $\varphi(y)$ *be O-regularly varying at* 0 *and such that for* $y \to 0$

(2.2) $\qquad y^{-r}\varphi(y)$ *almost decreases for some* $r>1$,

then, there exist two costants $\underline{l}>0$, $\bar{l}>\underline{l}$, *such that for every positive solution* $y(x)$ *of* (1.1) *tending to zero, there holds for* $x \geq x_0$

$$\underline{l}\left\{\int_a^x tf(t)\,dt\right\}^{-1} \leq \frac{\varphi(y(x))}{y(x)} \leq \bar{l}\left\{\int_a^x tf(t)\,dt\right\}^{-1}, \tag{2.3}$$

so that $y(x)$ *is O-regularly varying at infinity.*

By restricting the growth of $f(x)$ the integral in (2.3) can be disposed of; more precisely there holds [9]:

Corollary 1. *Let* $f(x)$ *be O-regularly varying at infinity and such that*

(2.4) $\qquad x^p f(x)$ *almost increases for some* $p<2$, *and let* $\varphi(y)$ *be O-regularly varying at* 0 *and such that for* $y \to 0$

(2.2) $$y^{-r}\varphi(y) \text{ almost decreases for some } r>1,$$

then there exist two constants $\underline{l}>0$, $\overline{l}>\underline{l}$ *such that for every positive solution* $y(x)$ *of* (1.1) *tending to zero, there holds for* $x \geqslant x_0$

(2.5) $$\underline{l}\{x^2 f(x)\}^{-1} \leqslant \frac{\varphi(y(x))}{y(x)} \leqslant \overline{l}\{x^2 f(x))\}^{-1}.$$

Notice here that, since in the condition (2.4) one has $p<2$, Corollary 1 does not cover e.g. the equation $y''=x^{-2}y^{\lambda}$, $\lambda>1$ where, in fact, $y(x) \sim c(\ln x)^{1/(1-\lambda)}$, $x \to \infty$[10, p. 150] more generally any of the equations (1.1) in which $x^2 f(x) = \psi(x)$ where $x^{\varepsilon}\psi(x) \to \infty$, $x^{-\varepsilon}\psi(x) \to 0$ for arbitrary $\varepsilon>0$.

Theorem 1, however, applies to that situation also. By assuming that $\psi(x)=L(x)$ — a slowly varying function, one obtains the following result which we shall need for deriving precise asymptotic formulae:

Corollary 2. *Let*

(2.6) $$\int_a^{\infty} t^{-1} L(t)\, dt = \infty$$

and let $\varphi(y)$ *be O-regularly varying at* 0, *and such that for* $x \to 0$

(2.2) $$y^{-r}\varphi(y) \text{ almost decreases for some } r>1,$$

then there exist two constants $\underline{l}>0$, $\overline{l}>\underline{l}$ *such that for every positive solution* $y(x)$ *tending to zero, of the equation*

$$y''=x^{-2}L(x)\varphi(y)$$

there holds for $x \geqslant x_0$

(2.7) $$\underline{l}\left\{\int_a^x t^{-1}L(t)\,dt\right\}^{-1} \leqslant \frac{\varphi(y(x))}{y(x)} \leqslant \overline{l}\left\{\int_a^x t^{-1}L(t)\,dt\right\}^{-1}.$$

The occuring integral cannot be disposed of in general, since it requires different treatments for different $L(t)$ due to (1.3).

2.2 *Asymptotic behavior.* If we restrict the class of considered equation by taking $f(x)$ to be o-regularly varying i.e. $f(x)=x^{\sigma}L(x)$ and $\varphi(y)=y^{\lambda}$, $\lambda>1$, we obtain the precise asymptotic behavior of solutions in question:

Theorem 2. *For every solution tending to zero, of the equation*

(2.8) $$y''=x^{\sigma}L(x)y^{\lambda}, \qquad \lambda>1,$$

there holds:

a) *For* $\sigma>-2$, $y(x)$ *is o-regularly varying at infinity and*

(2.9) $$y(x) \sim \{(1+\lambda+\sigma)(2+\sigma)(1-\lambda)^2\}^{\frac{1}{\lambda-1}}\{x^{2+\sigma}L(x)\}^{\frac{1}{\lambda-1}} x \to \infty,$$

b) *for* $\sigma=-2$, $y(x)$ *is slowly varying at infinity and*

$$y(x)\sim(\lambda-1)^{\frac{1}{1-\lambda}}\left\{\int_a^x t^{-1}L(t)\,dt\right\}^{\frac{1}{1-\lambda}},\quad x\to\infty. \tag{2.10}$$

This clarifies the behavior of such solutions of (2.8) completely, since these do not exist for $\sigma<-2$ due to the Wong's result.

Part a) i.e. (2.9) was proved by Avakumović [11]. Formulae (2.9) and (2.10) generalize corresponding results of Fowler where $L(x)=1$, cf. [10, Ch. 7].

3. Proofs. First notice that, by using the Proposition 3, one concludes that for the functions f and φ, in addition to (2.1) and (2.2) the following hold: For large x,

(3.1) $x^p f(x)$ almost increases for some p,

(3.2) $x^q f(x)$ almost decreases for some $q<p$,

for small y:

(3.3) $y^{-s}\varphi(y)$ almost increases for some $s>r>1$.

Notice also that solutions in question are convex due to the positivity of f and φ, and that $y'(x)$ is negative and tends to zero for $x\to\infty$.

All inequalities occuring in the proofs take place for $x\geqslant x_0$ which will be, therefore, occasionaly omitted. Furthermore, all minorizing constants will be denoted by the same letter m and similarly all majorizing ones by M, whenever possible. This is done to simplify the notation and because we are not interested in the actual values of the constants, though all these can be computed in terms of previously occuring ones.

3.1 *Proof of Theorem* 1. We first prove the inequality

$$\frac{\varphi(y(x))}{y(x)}\leqslant \bar{l}\left\{\int_a^x tf(t)\,dt\right\}^{-1},\quad x\geqslant x_0: \tag{3.4}$$

To that end, integrate (1.1) over (x, kx) with an arbitrary fixed $k>1$ and use $y'(x)<0$; this gives

$$-y'(x)\geqslant\int_x^{kx} f(t)\,\varphi(y(t))\,dt.$$

Or, for large x and hence for small y, because of (3.1) and (2.2),

$$-y'(x)\geqslant mx^p f(x)\,\varphi(y(kx))\int_x^{kx} t^{-p}\,dt.$$

Whence for $p\neq 1$

$$-y'(x)\geqslant mxf(x)\,\varphi(y(kx)) \tag{3.5}$$

holding for all $k>1$ and $x\geqslant x_0$. Notice that if it were $p=1$, we could take any $p>1$ since (3.1) a fortiori holds for any such p.

On the other hand, by multiplying (1.1) by $-y'(x)$ and integrating over (x, kx) one obtains

$$y'^2(x) \geqslant \int_x^{kx} f(t)\varphi(y(t))(-dy)$$

or, by (3.1) and (3.3)

$$y'^2(x) \geqslant mx^p f(x) y^{-s}(x) \varphi(y(x)) \int_x^{kx} x^{-p} y^s(-dy).$$

Whence, for all $k > 1$, and $x \geqslant x_0$

$$-y'(x) \geqslant m\{f(x)y(x)\varphi(y(x))\}^{1/2}\left\{1 - \left(\frac{y(kx)}{y(x)}\right)^{(s+1)/2}\right\}. \tag{3.6}$$

From (3.5) and (3.6) we shall derive the inequality

$$-y'(x) \geqslant mxf(x)\varphi(y(x)), \qquad x \geqslant x_0. \tag{3.7}$$

To that end we consider the quotient $K(x) = y(kx)/y(x)$ arising from (3.5) and occuring in (3.6). One has $0 < K(x) < 1$, hence the behavior of $K(x)$ is essential for the further use of (3.5) and (3.6). For, when e.g. $\lim_{k\to\infty} K(x) = 1$, (3.6) is useless. In order to be able to cope with all possible kinds of behavior of $K(x)$ we use the following Lemma on convex functions proved earlier by the authors [9], which is independent of differential equations.

Lemma. *Let $y(x)$ be a positive, continuous, convex function defined for $x > 0$, decreasing to zero for $x \to \infty$, and let $\{x_i\}$ be a sequence such that $x_i \to \infty$, $i \to \infty$. If for some $1 < k < 2$ and some $0 < r' < 1$ there holds*

$$\frac{y(kx_i)}{y(x_i)} < r'$$

then there exist numbers $0 < \mu < 1$, $0 < r < 1$, $k_1 > 1$ such that

$$\frac{y(k_1 x)}{y(x)} < r \text{ for all } x \in [\mu x_i, x_i].$$

Due to the continuity of $y(x)$ the following alternative holds (Cf. [9]):

Let $0 < r' < 1$, $1 < k < 2$, $x \geqslant x_0$, then:

Either

$$\frac{y(kx)}{y(x)} \geqslant r' \tag{3.8}$$

for all x belonging to some intervals I_n, $n = 1, 2, \ldots$, which may be all ultimately neighboring, when $\bigcup_{n=1}^{\infty} I_n = [a, \infty)$ for some $a \geqslant x_0$, or

$$\frac{y(kx)}{y(x)} < r' \tag{3.9}$$

for all x belonging to some intervals I_n^*, $n = 1, 2, \ldots$, which again may be all ultimately neighboring when $\bigcup_{n=1}^{\infty} I_n^* = [a, \infty)$, $a \geqslant x_0$.

If (3.8) holds, then by applying the condition (3.3) to the following form of the inequality (3.5)

$$-y'(x) \geq mxf(x)\frac{\varphi(y(kx))}{y^s(kx)}y^s(kx),$$

one obtains (3.7) for all $x \in I_n$, with some constant m' computed in terms of previous ones.

If, on the other hand (3.9) holds, we show first that for all $x \in I_n^*$

$$\frac{y(x)}{\varphi(y(x))} \geq mx^2 f(x). \tag{3.10}$$

To that end choose a sequence $\{x_n\}$ such that x_n is an arbitrary point of I_n^*, $n=1, 2, \ldots$, so that (3.9) holds for $x=x_n$. Then, because of the Lemma, there exist numbers $0<\mu<1$, $k_1>1$ such that

$$\frac{y(k_1 x)}{y(x)} < r' \quad \text{for all} \quad x \in [\mu x_n, x_n].$$

Hence, from (3.6) with $k=k_1$ and the above inequality, there follows

$$-y'(x) \geq m\{f(x)y(x)\varphi(y(x))\}^{1/2}, \quad x \in [\mu x_n, x_n], \tag{3.11}$$

or

$$\int_{\mu x_n}^{x_n} \left\{\frac{y^r}{\varphi(y)}\right\}^{1/2} y^{-(r+1)/2}(-dy) \geq m \int_{\mu x_n}^{x_n} (t^q f(t))^{1/2}\, t^{-q/2}\, dt. \tag{3.12}$$

The lefthand side of (3.12) is, by the use of (2.2), majorized by

$$M\left\{\frac{y^r(x_n)}{\varphi(y(x_n))}\right\}^{1/2} \int_{\mu x_n}^{x_n} y^{-(r+1)/2}(-dy).$$

Similarly the righthand side of (3.12) is, by the use of (3.2) with $q \neq 2$, minorized by

$$m(x_n{}^q f(x_n))^{1/2} \int_{\mu x_n}^{x_n} t^{-q/2}\, dt.$$

Hence, by integrating both of the above integrals, taking care of $r>1$ and $q \neq 2$, (3.12) is reduced to (3.10) with $x=x_n$. Since x_n is arbitrary in I_n^*, the inequality (3.10) is proved for all $x \in I_n^*$. Notice again, that if $q=2$ one could take any $q<2$ since (3.2) holds a fortiori for any such q. Now. from (3.11) which is valid for all $x \in I_n^*$, x_n being arbitrary in these intervals, and from (3.10) put in the equivalent form

$$(f(x)y(x)\varphi(y(x))^{1/2} \geq mxf(x)\varphi(y(x)),$$

one obtains again (3.7) for all $x \in I_n^*$, with some constant m'' computed in terms of previous ones. Consequently, the inequality (3.7) holds with $m=\min(m', m'')$

for all $x \geqslant a \geqslant x_0$ since one has $\bigcup_{n=1}^{\infty} I_n \cup I_n^* = [a, \infty)$ according to the definition of the intervals I_n, I_n^*.

To conclude the proof write (3.7) as

$$\int_a^x \frac{y^r}{\varphi(y)} y^{-r}(-dy) \geqslant m \int_a^x tf(t)\,dt \tag{3.13}$$

and, using (2.2) as before, majorize the lefthand side of (3.13), by $M \frac{y(x)}{\varphi(y(x))}$ which yields (3.4).

Next we prove the inequality

$$l\left\{\int_a^x tf(t)\,dt\right\}^{-1} \leqslant \frac{\varphi(y(x))}{y(x)}, \qquad x \geqslant x_0, \tag{3.14}$$

using again the alternative (3.8) — (3.9). Let (3.8) hold and take an arbitrary $x \in I_n$, then we proceed as in [9, p. 265]: Define $y_1(x)$ as follows: $y_1(x) = y(x)$, $x \in I_n$, $y_1(x) = c_n$, $x \notin I_n$, where the constant c_n coincides with the value of $y(x)$ in the righthand end of I_n, $n = 1, 2, \ldots$; see Fig. 1. Hence $y(x)$ is continuous outside of a countable set of points and $y_1'(x) = y'(x)$, $y_1''(x) = y''(x)$ for $x \in I_n$ and $y_1'(x) = y_1''(x) = 0$ elsewhere. Consequently due to (2.2) there holds the following inequality

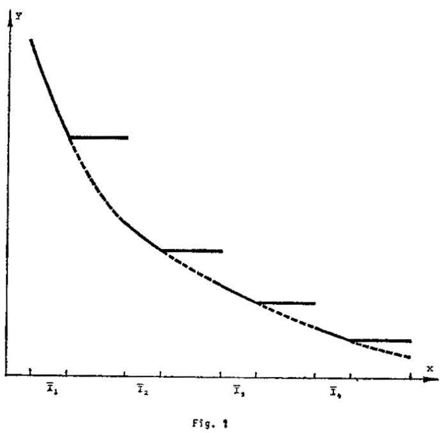

Fig. 1

$$y_1''(x) \leqslant Mf(x)\varphi(y_1(x)), \quad x \geqslant x_0. \tag{3.15}$$

Moreover, due to the definition of $y_1(x)$ the functions $y_1^{-r}\varphi(y_1)$ and $y_1^{-s}\varphi(y_1)$ also satisfy the conditions (2.2) and (3.3).

Take $x \in I_n$, and let firstly $kx \in I_n$; then using (2.2) and (3.2) as before one gets

$$y_1'(kx) - y_1'(x) \leqslant Mxf(x)\varphi(y_1(x)),$$

or by another integration over (a, x) for some $a \geqslant x_0$,

$$\int_a^x \frac{y_1'(kt) - y_1'(t)}{\varphi(y_1(t))}\,dt \leqslant M \int_a^x tf(t)\,dt. \tag{3.16}$$

In order to minorize the lefthand side of (3.16) write it as the difference of two integrals the first of which is minorized by (3.3) and the second majorized by (2.2); whence, the left hand side of (3.16) is minorized by

$$m\frac{y_1^s(x)}{\varphi(y_1(x))}\int_a^x\frac{y_1'(kt)}{y_1^s(t)}\,dt-M\frac{y_1^r(x)}{\varphi(y_1(x))}\int_a^x\frac{y_1'(t)}{y_1^r(t)}\,dt,$$

or, using (3.8) and integrating, by

$$\frac{y_1(x)}{\varphi(y_1(x))}\left\{\frac{M}{r-1}-\frac{m}{k(s-1)}\left(\frac{y_1(x)}{y_1(kx)}\right)^{s-1}+O(y_1^{r-1}(x))\right\}$$

and finally, due to (3.8) and since $1<r<s$, by $y_1(x)\varphi^{-1}(y(x))(M-m/k)$. Consequently by choosing k such that

(3.17) $$M-m/k\geqslant\eta>0$$

one obtains

(3.18) $$\int_a^x\frac{y_1'(kt)-y_1'(t)}{\varphi(y_1(t))}\,dt\geqslant m\frac{y_1(x)}{\varphi(y_1(x))}.$$

Inequalities (3.16) and (3.18) give together

(3.19) $$\frac{y_1(x)}{\varphi(y_1(x))}\leqslant M\int_a^x tf(t)\,dt$$

holding for $x\in I_n$ if also $kx\in I_n$. If, however, $kx\notin I_n$ then $y'(kx)=0$ so that (3.19) follows readily from (3.16). Since $y_1(x)=y(x)$ for $x\in I_n$, inequalities (3.19) and (3.14), with $l=M^{-1}$ coincide in the intervals I_n. We are left, therefore, with the proof of (3.14) in the intervals I_n^* in which there holds (3.9).

As in the previous case integrate (1.1) over (x, kx) $k>1$ and make use of (3.2) and (3.3); this leads to

$$y'(kx)-y'(x)\leqslant Mxf(x)\varphi(y(x)),$$

and by repeating the procedure to

$$y(x)-\left(1+\frac{1}{k}\right)y(kx)\leqslant M\varphi(y(x))\int_x^{kx}t^qf(t)\,t^{1-q}\,dt.$$

Using (3.9) and (3.2) with $q\neq 2$, one obtains

(3.20) $$y(x)\left\{r'\left(1+\frac{1}{k}\right)-\frac{1}{k}\right\}\leqslant Mx^2f(x)\varphi(y(x)$$

which is reduced to

(3.21) $$y(x)\leqslant Mx^2f(x)\varphi(y(x)),\qquad x\geqslant x_0,$$

by choosing k such that — in addition to (3.17) — $r'(1+1/k)-1/k\geqslant\eta>0$. Since, because of (3.2), the relation

(3.22) $$x^2f(x)\leqslant M\int_a^x tf(t)\,dt,\qquad x\geqslant x_0$$

is obvious, the wanted inequality (3.14) for $x \in I_n^*$ follows from (3.21). This completes the proof of Theorem 1.

3.2 *Proof of Corollary* 1. Since $p<2$ the condition (2.1) is fulfilled and the Theorem 1 applies. Furthermore (3.22) holds so that for the proof only the inequality

$$\int_a^x tf(t)\,dt \leqslant Mx^2 f(x), \qquad x \geqslant x_0 \tag{3.23}$$

is still needed. But, because of (2.4)

$$\int_a^x tf(t)\,dt \leqslant Mx^p f(x) \int_a^x t^{-p+1}\,dt \leqslant Mx^2 f(x).$$

Now (2.3), (3.22) and (3.23) together give (2.5) qed.

3.3 *Proof of Corollary* 2. The function $f(x)=x^{-2}L(x)$ is O-regularly varying at infinity because of Proposition 2 and 3. Since all other conditions of the Theorem 1 are fulfilled by hypothesis it applies again giving (2.8).

3.4 *Proof of Theorem* 2. We have to prove only part b) since, as we mentioned in the introduction, part a) has already been proved.

We first show that $y(x)$ is slowly varying and then apply Proposition 1.

Put $L_1(x)=\left\{\int_a^x t^{-1}L(t)\,dt\right\}^N, \qquad N>0.$

The occuring integral diverges due to the Wong's result so that the slowly varying function L_1 (Cf. 1.3) tends to infinity with x. By differentiating $y(x)L_1(x)$, using $y'(x)=-\int_a^\infty t^{-2}L(t)y^\lambda(t)\,dt$ and the righthand inequality in (2.7) with $\varphi(y)=y^\lambda$, one obtains for $x \geqslant x_0$

$$(y(x)L_1(x))' \geqslant x^{-1}y(x)L(x)\left(\int_a^x t^{-1}L(t)\,dt\right)^{N-1}\{N-\overline{l}^{\lambda-1}\}.$$

By taking large N, it follows that the above derivative is positive so that $y(x)L_1(x)$ increases. Hence for $x \geqslant x_0$ and $k>1$ (similarly for $k<1$) one has $y(x)L_1(x) \leqslant y(kx)L_1(kx)$ and so, $y(x)$ being decreasing and $L(x)$ slowly varying,

$$1>\frac{y(kx)}{y(x)} \geqslant \frac{L_1(x)}{L_1(kx)}>1-\varepsilon, \quad x \geqslant x_0.$$

Therefore $y(x)$ is slowly varying according to the Definition 2.

Now put $y'(x)$ in the form

$$-y'(x)=x^{-1}\int_1^\infty t^{-2}L^*(xt)\,dt$$

where $L^*(t)=L(t)y^\lambda(t)$ is a slowly varying function as a product of such ones and apply the Proposition 1; this gives $-y'(x)\sim x^{-1}L(x)y^\lambda(x)$, $x\to\infty$. By dividing through $y^\lambda(x)$ and integrating over (a, x) one obtains (2.11) bearing in mind that $\int_a^\infty t^{-1}L(t)dt\to\infty$, $x\to\infty$. This completes the proof of Theorem 2.

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