PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 21 (35) 1977, pp 9–14

SOME RESULTS ON ĆIRIĆ'S QUASI—CONTRACTION MAPPINGS

J. Achari

(Received July 30, 1976)

Introduction.

Recently Ćirić [2] proved some fixed point theorems when the mapping f of a metric space X into itself satisfies the following inequality

(1) $$d(fx, fy) \leqslant \alpha \max \{d(x, y),\ d(x, fx),\ d(y, fy),\ d(x, fy),\ d(y, fx)\}$$

for $x, y \in X$ and for some α, $0 < \alpha < 1$. He termed this type of mapping as quasi-contraction mapping. Ćirić has shown that if f satisfies (1) in a complete (orbitally complete) metric space, then f has a unique fixed point.

This paper is devoted to the study of fixed points of quasi-contraction mappings in metric and normed spaces. In section **1**, we have studied the behaviour of a sequence of fixed points with the convergence of corresponding quasi-contraction mappings. In section **2** we have discussed Reinermann's [7] iteration scheme in Banach and Hilbert spaces converging to the fixed point of a quasi-contraction mapping.

1. Sequence of quasi-contraction mappings.

Theorem 1. *Let (x, d) be a metric space. Let $f_i : X \to X$ be a mapping with at least one fixed point z_i for each $i = 1, 2, 3, \ldots$ and $f_0 : X \to X$ be a mapping satisfying* (1) *with fixed point z_0. If the sequence $\{f_i\}$ converges uniformly to f_0 then the sequence $\{z_i\}$ converges to z_0.*

Proof. Choose an arbitrary positive number $\dfrac{\varepsilon(1-\alpha)}{(1+\alpha)}$. Corresponding to this choice we can find a positive integer N such that $d(f_i z_i, f_0 z_i) < \dfrac{\varepsilon(1-\alpha)}{(1+\alpha)}$ for $i \geqslant N$, since f_i converges uniformly to f_0.

Now

$$d(z_i, z_0) = d(f_i z_i, f_0 z_0) \leqslant d(f_i z_i, f_0 z_i) + d(f_0 z_i, f_0 z_0) \leqslant d(f_i z_i, f_0 z_i) +$$
$$+ \alpha \max \{d(z_i, z_0),\ d(z_i, f_0 z_i),\ d(z_0, f_0 z_0)\ d(z_i, f_0 z_0),\ d(z_0, f_0 z_i)\} \leqslant$$
$$\leqslant d(f_i z_i, f_0 z_i) + \alpha [d(f_0 z_i, z_i) + d(f_i z_0, z_i)] \leqslant \frac{1+\alpha}{1-\alpha} d(f_i z_i, f_0 z_i)$$

Hence we get $d(z_0, z_i) < \varepsilon$ which is required.

Theorem 2. *Let (X, d_n) be a metric space for each $n=0, 1, 2, \ldots$ and suppose that $\{d_n\}_{n=0}^{\infty}$ converges uniformly to d_0. Let $f_n:(X, d_n)\to(X, d_n)$ satisfy* (1) *for all $n=1, 2, \ldots$ and with fixed points z_n. If $f_0:(X, d_0)\to(X, d_0)$ can be defined as the d_0 pointwise limit of f_n, then $z_n \underset{d_0}{\to} z$, the unique fixed point of f_0.*

Proof. First we shall show that f_0 satisfies (1) with respect to d_0. Now

$$d_0(f_0 x, f_0 y)\leqslant d_0(f_0 x, f_n x)+d_0(f_n x, f_n y)+d_0(f_n y, f_0 y)$$

Since the latter inequality is valid for $n\geqslant N$, we have

$$d_0(f_0 x, f_0 y)\leqslant d_0(f_0 x, f_n x)+d_n(f_n x, f_n y)+\varepsilon+d_0(f_n y, f_0 y)$$

$$\leqslant d_0(f_0 x, f_n x)+\alpha\max\{d_n(x, y), d_n(x, f_n x), d_n(y, f_n y), d_n(x, f_n y), d_n(y, f_n x)\}+\varepsilon+d_0(f_n y, f_0 y)$$

$$\leqslant d_0(f_0 x, f_n x)+\alpha\max\{d_0(x, y)+\varepsilon, d_0(x, f_n x)+\varepsilon, d_0(x, f_n y)+\varepsilon, d_0(y, f_n x)+\varepsilon\}+\varepsilon+d_0(f_n y, f_0 y)$$

$$\xrightarrow[n\to\infty]{} \alpha\max\{d_0(x, y), d_0(x, f_0 x), d_0(y, f_0 y), d_0(x, f_0 y), d_0(y, f_0 x)\}$$

Since this is true for every $\varepsilon>0$ we get the desired conclusion.

Now choose an arbitrary positive number $\frac{\varepsilon(1-\alpha)}{1+\alpha}$. Corresponding to this choice we can find a positive integer N such that $\varepsilon=\frac{\varepsilon(1-\alpha)}{2(1+2\alpha)}$ and $d_0(f_n z_0, f_0 z_0)<\frac{\varepsilon(1-\alpha)}{2(1+\alpha)}$ for $n\geqslant N$, since f_n converges uniformly to f_0.

Then

$$d_0(z_n, z_0)=d_0(f_n z_n, f_0 z_0)\leqslant d_0(f_n z_n, f_n z_0)+d_0(f_n z_0, f_0 z_0)$$

Since the latter inequality is valid for $n\geqslant N$, we have

$$d_0(z_n, z_0)\leqslant d_n(f_n z_n, f_n z_0)+\varepsilon+d_0(f_n z_0, f_0 z_0)$$

$$\leqslant\alpha\max\{d_n(z_n, z_0), d_n(f_n z_n, z_n), d_n(z_0, f_n z_0), d_n(z_n, f_n z_0), d_n(z_0, f_n z_n)\}+\varepsilon+$$

$$+d_0(f_n z_0, f_0 z_0)\leqslant\alpha[d_n(f_n z_0, z_0)+d_n(z_n, z_0)]+\varepsilon+d_0(f_n z_0, f_0 z_0)\leqslant$$

$$\leqslant\alpha[d_0(f_n z_0, f_0 z_0)+\varepsilon+d_0(z_n, z_0)+\varepsilon]+\varepsilon+d_0(d_0(f_n z_0, f_0 z_0)$$

$$\leqslant\frac{1+\alpha}{1-\alpha}d_0(f_n z_0, f_0 z_0)+\frac{1+2\alpha}{1-\alpha}\varepsilon$$

We get $d_0(z_n, z_0)<\varepsilon$ and hence the proof.

Theorem 3. *Let* (X, d) *be a complete metric space and* $f_n (n = 1, 2, 3)$ *be a mapping of* X *into itself satisfying* (1) *and with fixed points* z_n. *Suppose that a mapping* f *of* X *into itself can be defined by* $fx = \lim_{n\to\infty} f_n x$ *(for all* x*), then* $z = \lim_{n\to\infty} z_n$ *is the unique fixed point of* f.

Proof. Since $d: X \times X \to$ the reals is continuous, we immediately see that f satisfies (1) and therefore has a unique fixed point z. Now choose an arbitrary positive number $\frac{\varepsilon(1-\alpha)}{1+\alpha}$. Corresponding to this choice we can find a positive integer N such that $d(f_n z, fz) < \frac{\varepsilon(1-\alpha)}{1+\alpha}$ for $n \geqslant N$, since f_n converges uniformly to f_0.

Now we have

$$\begin{aligned} d(z_n, z) &= d(f_n z_n, fz) \leqslant d(f_n z_n, f_n z) + d(f_n z, fz) \\ &\leqslant d(f_n z, fz) + \alpha \max\{d(z_n, z), d(z_n, f_n z_n) \\ &\quad d(z, f_n z), d(z_n, f_n z), d(z, f_n z_n)\} \\ &\leqslant d(f_n z, fz) + \alpha [d(f_n z, z) + d(z_n, z)] \\ &\leqslant \frac{1+\alpha}{1-\alpha} d(f_n z, fz). \end{aligned}$$

Thus we get $d(z_n, z) < \varepsilon$ and this completes the pıoof of the theorem. Theorems 1, 2 and 3 generalize the results of Bonsall [1] and Nadler [5].

2. Fixed point iterations using infinite matrices and quasicontraction mappings

Let X be a normed linear space, M a non-empty closed bounded convex subset of X, f a mapping of M into itself possessing at least one fixed point in M and $A = (a_{nk})$ be an infinite matrix. Define the iteration scheme

(2) $$\bar{x}_0 = x_0 \in M$$

(3) $$\bar{x}_{n+1} = f x_n, \; n = 0, 1, 2, \ldots$$

(4) $$x_n = \sum_{k=0}^{n} a_{nk} \bar{x}_k, \; n = 1, 2, \ldots$$

The scheme (2)—(4) is generally known as Mann [4] process.

Recently Reinermann [7] has defined a summability matrix A by

(5) $$\begin{aligned} a_{nk} &= c_k \prod_{j=k+1}^{n} (1 - c_j), \quad k < n, \\ &= c_n \qquad\qquad k = n, \\ &= 0 \qquad\qquad k > n \end{aligned}$$

where the real sequence $\{c_n\}$ satisfies (i) $c_0=1$ (ii) $0<c_n<1$ and (iii) $\sum_k c_k$ diverges. He then defined the iteration scheme (2) and $x_{n+1}=\sum_{k=0}^{n} a_{nk} fx_k$, $x_0 \in M$ which can be written in the form

(6) $$x_n+1=(1-c_n)x_n+c_n fx_n$$

The same iteration scheme has been defined independently by Outlaw and Groetsch [6] and Dotson [3]. We point out however, that even the matrices involved are the same, iteration schemes (2)—(4) and (2), (5) are different.

Scheme (2)—(4) takes the form $x=Az$, where $z=\{x_0, fx_0, fx_1, \ldots\}$ whereas (2) and (5) become $x=A\omega$, where $\omega=\{fx_0, fx_1, \ldots\}$. Because the iteration scheme defined by (2), (5) is notionally simpler, the theorems of this paper will be stated and proved in terms of a matrix defined by (5) with $\{c_n\}$ satisfying) (i)—(iii). In this section we shall show that for particular spaces and for function f satisfying the inequality

(7) $$\|fx-fy\| \leqslant \alpha \max\{\|x-y\|, \|x-fx\|, \|y-fy\|, \|x-fy\| \, \|y-fx\|\}$$

for $x, y \in X$ and $0<\alpha<1$, the iteration scheme defined by (2), (5) converges to a fixed point of f.

Theorem 4. *Let X be a Banach space, $f: X \to X$ be a mapping of X into itself and satisfies* (7). *Suppose that A is an infinite matrix defined by* (5) *with $\{c_n\}$ satisfying* (i), (ii) *and bounded away from zero. If the sequence, defined by* (6) *converges to z then z is the unique fixed point of f in X.*

Proof. For each n, $x_{n+1}-x_n=c_n(fx_n-x_n)$. Since

$$\lim_{n\to\infty}=x_n=z, \quad \lim_{n\to\infty}\|x_{n+1}-x_n\|=0 \text{ Then } \lim_{n\to\infty}\|fx_n-x_n\|=0,$$

because $\{c_n\}$ is bounded away from zero. From (7) we get

$$\|fx_n-fz\| \leqslant \alpha \max\{\|x_n-z\|, \|x_n-fx_n\|, \|z-fz\|, \|x_n-fz\|, \|z-fx_n\|\}$$
$$\leqslant \alpha[\|fx_n-fz\|+\|fx_n-x_n\|+\|x_n-z\|]$$

which implies

$$\|fx_n-fz\| \leqslant \frac{\alpha}{1-\alpha}[\|fx_n-x_n\|+\|x_n-z\|]$$

Since the right hand side tends to zero as $n\to\infty$, then $fx_n \to fz$. Now,

$$\|fz-z\| \leqslant \|fz-fx_n\|+\|fx_n-x_n\|+\|x_n-z\| \to 0.$$

as $n\to\infty$, whence we get $fz=z$. This is also unique because of (7) this completes the proof.

Theorem 5. *Let E be a closed convex subset of a Hilbert space H and f be a self map of E satisfying (7). Let A be an infinite matrix defined by* (5) *with* $\{c_n\}$ *satisfying* (i)—(iii) *and* $\overline{\lim_n}\, c_n < 1-\alpha^2$. *Then the iteration scheme* (6) *converges to the fixed point of f.*

Proof. Since f satisfies (7), it has a unique fixed point $z \in H$. Let $p\cdot q \in H$ and λ, μ be two non-negative real numbers with $\lambda+\mu=1$. Now we can expand and add the inner products $(\lambda p+\mu q, \lambda p+\mu q)$ and $(\lambda(p-q), \mu(p-q))$ to obtain the identity

$$\|\lambda p+\mu q\|^2=\lambda\|p\|^2+\mu\|q\|^2-\lambda\mu\|p-q\|^2 \tag{8}$$

From the expression (6) we obtain

$$x_{n+1}-z=(1-c_n)(x_n-z)+c_n(fx_n-z)$$

whence, by applying (8), we derive

$$\|x_{n+1}-z\|^2=(1-c_n)\|x_n-z\|^2+c_n\|fx_n-z\|^2-c_n(1-c_n)\|fx_n-x_n\|^2 \tag{9}$$

Since f satisfies (7) we get

$$\begin{aligned}\|fx_n-fz\| &\leqslant \alpha\max\{\|x_n-z\|, \|x_n-fx_n\|, \|z-fz\|, \|x_n-f_z\|, \|z-fx_n\|\}\\ &\leqslant \alpha\max\{\|x_n-z\|, \|x_n-fx_n\|\}\end{aligned} \tag{10}$$

If in (10) the maximum is $\|x_n-z\|$, then from (9) we have

$$\|x_{n+1}-z\|^2\leqslant(1-c_n)\|x_n-z\|^2+c_n\alpha^2\|x_n-z\|^2-c_n(1-c_n)\|fx_n-x_n\|^2$$

and if the maximum is $\|x_n-fx_n\|$ then

$$\|x_{n+1}-z\|^2\leqslant(1-c_n)\|x_n-z\|^2+c_n\alpha^2\|x_n-fx_n\|^2-c_n(1-c_n)\|+x_n-x_n\|^2$$

In either case we get

$$\|x_{n+1}-z\|^2\leqslant\|x_n-z\|^2-c_n(1-c_n-\alpha^2)\|fx_n-x_n\|^2 \tag{11}$$

From (11) we observe that $\{\|x_{n+1}-z\|\}$ decreases with n for all n sufficiently large. Also, since $\{c_n\}$ satisfies (iii) and $\overline{\lim_n}\, c_n<1-\alpha^2$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|fx_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \tag{12}$$

Since f satisfies (7) we have

$$\|fx_{n_k}-x_{n_l}\|\leqslant\alpha\max\{\|x_{n_k}-x_{n_l}\|, \|x_{n_k}-fx_{n_k}\|, \|x_{n_l}-fx_{n_l}\|, \|x_{n_k}-fx_{n_l}\|, \|x_{n_l}-fx_{n_k}\|\}$$

implying

$$\|fx_{n_k}-fx_{n_l}\|\leqslant\frac{\alpha}{1-\alpha}[\|x_{n_k}-fx_{n_k}\|+\|x_{n_l}-fx_{n_l}\|] \tag{13}$$

Thus the sequence $\{fx_{n_k}\}$ is Cauchy and hence convergent. Let its limit be ω, so that $\lim_{k\to\infty} fx_{n_k}=\lim_{k\to\infty} x_{n_k}=\omega$. Now

$$(14)\qquad \|f\omega-\omega\|\leqslant\|f\omega-fx_{n_k}\|+\|x_{n_k}-x_{n_k}\|+\|x_{n_k}-\omega\|$$

and since f satisfies (7) we derive

$$(15)\qquad \begin{aligned}\|f\omega-fx_{n_k}\|&\leqslant\alpha\max\{\|\omega-x_{n_k}\|,\ \|\omega-f\omega\|,\ \|x_{n_k}-fx_{n_k}\|,\ \|\omega-fx_{n_k}\|,\ \|x_{n_k}-f\omega\|\}\\ &\leqslant\alpha[\|\omega-x_{n_k}\|+\|fx_{n_k}-x_{n_k}\|+\|f\omega-\omega\|]\end{aligned}$$

Substituting (15) into (16), we obtain

$$\|f\omega-\omega\|\leqslant\frac{1+\alpha}{1-\alpha}[\|\omega-x_{n_k}\|+\|fx_{n_k}\|]\to 0 \text{ as } k\to\infty$$

This implies $f\omega=\omega$. Since f has a unique fixed point z, then $z=\omega$. Because of the fact $\lim_{k\to\infty} x_{n_k}=z$ and that $\{\|x_n-z\|\}$ decreases with n for all sufficiently large n we get $\lim_{n\to\infty} x_n=z$, This completes the proof.

Acknowledgement. The author is thankful to Prof. B.E. Rhoades for his valuable support and interest.

REFERENCES

[1] F. F. Bonsall, *Lectures on some fixed point theorems of functional analysis*, Tata Institute of Fundamental Research, Bombay 1962.

[2] LJ. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. 45 (1974), 267—273.

[3] W. G. Dotsor Jr, *On the Mann iterative process*, Trans. Amer. Math. Soc. 149 (1970), 65—73.

[4] W. R. Mann, *Mean value methods in iteration*, Proc., Amer. Math. Soc. 4 (1953) 506—510.

[5] S. B. Nadler Jr, *Sequences of contractions and fixed points*, Pacific J. Math. 27 (1968), 579—585.

[6] C. L. Outlaw and C. W. Groetsch, *Averaging iteration in a Banach space*, Bull. Amer. Math. Soc. 75 (1969), 430—432.

[7] J. Reinermann, *Über Toeplitzsch Iterationverfahren und einige ihre Anwendungen in der konstruktiven Fixpunktheorie*, Studia Math. 32 (1969), 209—227.

[8] Rhoades, B. E., *Fixed point iterations using infinite matrices*, Trans. Amer. Math. Soc. 196 (1975), 161—176.

Department of Mathematics
Indian Institute of Technology
Kharagpur, India