PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 21 (35), 1977, pp. 115—118

REMARKS ON SOME FIXED POINT THEOREMS

M. Maiti, J. Achari and T. K. Pal

The purpose of the present note is to establish the generalizations of two fixed point theorems, one due to Meir and Keeler [1] and the other due to Boyd and Wong [2]. The mappings, discussed by these authors, are non-expansive and continuous. Here the mappings are also of the non-expansive type, but need not be continuous. The arguments, put forward in deriving the generalizations, are close in spirit to those given in the proofs of the original theorems.

The following theorem is due to Meir and Keeler.

Theorem 1 (Meir and Keeler). *Let* (x, ρ) *be a complete metric space and the mapping* $A: X \to X$ *be weakly uniformly strict contraction, i. e. given an* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\varepsilon \leqslant \rho(x, y) < \varepsilon + \delta \Rightarrow \rho(Ax, Ay) < \varepsilon \tag{1}$$

for $x, y \in X$. *Then A has a unique fixed point* $\xi \in X$ *and, for each* $x \in X$, $A^n x \to \xi$ *as* $n \to \infty$.

In Theorem 1 the mapping A is continuous, because it is obviously contractive. We now consider a mapping A which satisfies

$$\varepsilon \leqslant \max\{\rho(x, y),\ \rho(x, Ax),\ \rho(y, Ay)\} < \varepsilon + \delta \Rightarrow \rho(Ax, Ay) < \varepsilon \tag{2}$$

instead of (1). In this case A may not be continuous. We prove here the existence of a fixed point of this type of mapping with a further assumption that the functional $F(x) = \rho(x, Ax)$ is lower semicontinuous.

The relation (2) implies that

$$\rho(Ax, Ay) < \max\{\rho(x, y),\ \rho(x, Ax),\ \rho(y, Ay)\} \tag{3}$$

for $x \neq y$. Construct a sequence $\{x_n\}$ such that $x_n = A^n_x$, where $x \in X$. Then from (3) we derive that $\rho(x_{n+1}, x_{n+2}) < \rho(x_n, x_{n+1})$. Hence $\rho(x_n, x_{n+1})$ decreases with n and, because of (2), $\rho(x_n, x_{n+1} \to 0$ as $n \to \infty$. Next, we show that the sequence $\{x_n\}$ is fundamental. If it is not, there exists an $\varepsilon > 0$ such that $\limsup \rho(x_m, x_n) > 2\varepsilon$. Since $\rho(x_n, x_{n+1}) \to 0$, there exists a positive integer M such that $\rho(x_j, x_{j+1}) < \delta'/3$ for $j \geqslant M$, where $\delta' = \min(\delta, \varepsilon)$. Choose further two integers $m, n > M$ so that $\rho(x_m, x_n) > 2\varepsilon > \varepsilon + \delta'$. This implies that there exists an integer k in $[m, n]$ with

$$\varepsilon + \frac{2\delta'}{3} < \rho(x_m, x_k) < \varepsilon + \delta'. \tag{4}$$

Now, from the triangle inequality we obtain

$$\begin{aligned} \rho(x_m, x_k) &\leqslant \rho(x_m, x_{m+1}) + \rho(x_{m+1}, x_{k+1}) + \rho(x_{k+1}, x_k) \\ &< \frac{\delta'}{3} + \varepsilon + \frac{\delta'}{3} = \varepsilon + \frac{2\delta'}{3} \end{aligned} \tag{5}$$

since $\max\{\rho(x_m, x_k), \rho(x_m, x_{m+1}), \rho(x_k, x_{k+1})\} = \rho(x_m, x_k)$. Then (5) contradicts (4) and hence $\{x_n\}$ is a fundamental sequence. Let $\xi = \lim_{n\to\infty} x_n$. Since $F(x)$ is lower semicontinuous, $\liminf F(x_n) \geqslant F(\xi)$. But $\rho(x_n, x_{n+1}) \to 0$ as $n \to \infty$; thus $F(\xi) = 0$, implying $A\xi = \xi$. Then ξ is a fixed point of A and, because of (3), it is obviously unique. Thus we have established the following:

Theorem 2. *Let (X, ρ) be a complete metric space and $A: X \to X$ be a mapping satisfying* (2). *If the functional $F(x) = \rho(x, Ax)$ is lower semicontinuous, then A has a unique fixed point.*

We note that a contractive mapping satisfies (3), which has been introduced by Sehgal [3] in order to prove the existence of a fixed point of a compact space. It may be further noted that the condition (2) is weaker than the condition (1). This is illustrated by the following example.

Example 1. Let $X = [0,7]$ and the function f be defined by

$$\begin{aligned} f(x) &= \frac{2}{5}x, \quad 0 \leqslant x \leqslant 6; \\ &= -2x + 14, \quad 6 < x \leqslant 7. \end{aligned}$$

In this case (1) is not satisfied, since f is discontinuous at $x = 6$. But the condition (2) is satisfied and $F(x)$ is also lower semicontinuous. Thus Theorem 2 is applicable.

Theorem 3 (Boyd and Wong). *Let (X, ρ) be a complete metric space and let the mapping $A: X \to X$ satisfy*

$$\rho(Ax, Ay) \leqslant \Phi(\rho(x, y)) \tag{6}$$

where Φ; $\bar{P} \to [0, \infty)$ is an upper semicontinuous function from the right on $\bar{P}$ $[P = \{\rho(x, y) \mid x, y \in X$; $\bar{P} =$ closure of $P]$ and satisfies $\Phi(t) < t$ for all $t \in \bar{P}\{0\}$. Then A has a unique fixed point $\xi \in X$ and $A^n x \to \xi$ for each $x \in X$.

Theorem 3 may be generalized in the following manner:

Theorem 4. *Let (X, ρ) be a complete metric space and A be a self-mapping of X such that*

(i) $\rho(Ax, Ay) \leqslant \max\{\Phi(\rho(x, y)), \Phi(\rho(x, Ax)), \Phi(\rho(y, Ay))\}$

where $\Phi:\bar{P}\to[0,\infty)$ *is an upper semicontinuous function from the right on* $\bar{P}$ *satisfying* $\Phi(t)<t,\ t>0$ *and* $\Phi(0)=0$; (ii) *the functional* $F(x)=\rho(x, Ax)$ *is lower semicontinuous.*

Then A *has a unique fixed point* $\xi\in X$ *and* $A^n x\to\xi$ *for each* $x\in X$.

Proof. Constructing a sequence of iterates $\{x_n\}$ as before and setting $c_{n+1}=\rho(x_n, x_{n+1})$ we derive from (7)

$$c_{n+1}\leqslant\Phi(c_n)<c_n, \tag{8}$$

implying that c_n decreases with n. Because of the upper semicontinuity of Φ it is easy to show that $c_n\to 0$ as $n\to\infty$.

We now show that the sequence $\{x_n\}$ is fundamental. Let us assume that it is not so. Then there exist an $\varepsilon>0$ and sequences of integers $\{m(k)\}$, $\{n(k)\}$ with $m(k)>n(k)\geqslant k$ such that

$$d_k=\rho(x_{m(k)}, x_{n(k)})\geqslant\varepsilon,\ k=1, 2, 3, \dots . \tag{9}$$

If $m(k)$ is the smallest integer exceeding $n(k)$ for which (9) holds, then, from the well-ordering principle, we have

$$\rho(x_{m(k)-1}, x_{n(k)})<\varepsilon.$$

Now

$$d_k=\rho\, x_{m(k)}, x_{n(k)})\leqslant\rho(x_{m(k)}, x_{m(k)-1})+\rho(x_{m(k)-1}, x_{n(k)})$$

$$<c_{m(k)}, +\varepsilon<c_k+\varepsilon.$$

This implies that $d_k\to\varepsilon$ as $k\to\infty$. Also, we have

$$d_k\leqslant\rho(x_{m(k)}, x_{m(k)+1})+\rho(x_{m(k)+1}, x_{n(k)})+\rho(x_{n(k)+1}, x_{n(k)})$$

$$\leqslant c_{m(k)+1}+c_{n(k)+1}+\max\{\Phi(d_k),\ \Phi(c_{m(k)+1}),\ \Phi(c_{n(k)+1})\}. \tag{10}$$

Letting $k\to\infty$ in (10), we obtain

$$\varepsilon\leqslant\Phi(\varepsilon)>\varepsilon,$$

which is impossible, Thus the sequence $\{x_n\}$ is fundamental and the limit of the sequence is the unique fixed point of A.

It is to be noted that the condition (6), with $\Phi(t)<t$, implies (7). The following example will establish that Theorem 4 is more general than Theorem 3.

Example 2. Let $X=[0,7]$ and a function f be defined on X as

$$f(x)=\frac{x}{2},\ 0\leqslant x\leqslant 6;$$

$$=-2x+14,\ 6<x\leqslant 7.$$

We also define an upper semicontinuous function Φ (from the right) as

$$\Phi(t)=\frac{4}{5}t,\ 0\leqslant t<6;$$

$$=\frac{3}{5}t,\ 6<t\leqslant 7.$$

Theorem 3 is not applicable to this example, whereas all the conditions of Theorem 4 are satisfied here.

REFERENCES

[1] A. Meir and Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. 28 (1969), 326—329.

[2] D. W. Boyd and J. S. W. Wong, *On non-linear contractions*, Proc. Amer. Math. Soc. 28 (1969), 458—464.

[3] V. M. Sehgal, *On fixed and periodic points for a class of mappings*, J. Lond. Math. Soc. 5 (1972), 571—576.

Indian Institute of Technology,
Kharagpur, INDIA

Kharagpur College,
Kharagpur INDIA