PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 21 (35), 1977, pp. 99—108

ON SOME HYPOTHESES CONCERNING TREES[1)]

Kurepa, Đuro (Beograd)

0. We consider several statements concerning trees, in particular two statements on trees of linear measurable sets; in special cases both statements are true, but in general case they are independent of usual axioms of the theory of sets, and are connected with postulates on ramifications. We emphasize two alternatives: *TA* (tree alternative, v. § 3 : 1) and *GTA* (general tree alternative) respectively; cf § 3 : 2). — two equivalent postulates (s. 4, 7 : 10; also: Théorème fondamental p. 132 in Đ. KUREPA (1935)).

1. Statement *P*. *Every decreasing infinite tree* $(T, \supset)$ *of linear measurable sets is D-reflexive* (for the terminology see KUREPA, Đ. (1974)).

1 : 1. Remark. If in the tree $(T, \supset)$ there exists a strictly decreasing real function, then the statement P is true (see KUREPA, Đ. (1937), (1940—41), Theor. 2, Theor. 8.). But if such a function does not exist, then the statement P is not provable[2].

1 : 2. Theorem. *The statement P is equivalent to the following statement:*
1 : 2 : 1. *P'. Every infinite tree of cardinality* $\leqslant 2^{\aleph_0}$ *is D-reflexive.*

Proof. The implication $P' \Rightarrow P$ being obvious, let us prove that $P \Rightarrow P'$. Now, if T is any infinite tree of cardinality $\leqslant 2^{\aleph_0}$, let us consider the triadic set Tr of all real numbers $R[0, 1]$ representable in the positional number system with basis 3 and without a use of the digit 1; Tr is perfect of cardinality $2^{\aleph_0}$ and of measure 0; let f be any one-to-one mapping of the tree T into the set Tr. For every

$t \in T$ let $g(t) := \{fx \mid x \in T[t, .) := \{y \mid y \in T,\ t \leqslant y\}\}$. Let $g(T) := \{g(t) \mid t \in T\}$; then

$(g(T), \supset)$ is a ramified decreasing tree of sets; the mapping $g: T \to g(T)$ is an isomorphism between the trees $(T, \leqslant)$, $(g(T), \supset)$. Now the elements of $g(T)$ are subsets of Tr and consequently are measurable (of measure 0); by hypothesis P the tree $(g(T), \supset)$ is D-reflexive; so is $(T, \leqslant)$ as well, the correspondence g/T being an isomorphism between the trees $(T, \leqslant)$, $(g(T), \supset)$.

[1)] Presented the 1975. 08. 29. 5. at the 5th Congress of mathematicians, physicists and astronomers of Yugoslavia (Novi Sad, 1975. 08. 28. 4. — 09. 02. 2.).

[2)] Cf. the propositions P_1—P_{12} in KUREPA, Đ. (1935) p. 130—131; cf. (1968) and (1974).

2. On the other hand, the number $2^{\aleph_0}$ could be assumed to be any non countable cardinal number of non countable cofinality (announced in KUREPA Đ. 1953 b, proved in EASTON, W. B. 1970). Therefore the statement P is equivalent to the following.

2:1. Statement P''. *Let α be any fixed ordinal number; every ordered chain $(L, \leqslant)$ of a cardinality $\leqslant\aleph_\alpha$ satisfies $s(L, \leqslant)=c(L, \leqslant)$.*

It is instructive to compare the statement P'' and the following

2:2. Statement P^0: Or $(\alpha)\Rightarrow((cL\leqslant\aleph_\alpha)\Rightarrow(sL\leqslant\aleph_\alpha))$.

The last statement is equivalent to the following one:

2:3. P_5^0 *Every infinite ordered chain L satisfies $cL=sL$.*

Now, we considered the following:

2:4. Statement P_5: *For every infinite linearly ordered set L there exists a disjoint family of cardinality sL of non empty intervals of L* (see Đ. KUREPA (1935) p. 131).

One can prove the following

2:5. Theorem. $P_5\Leftrightarrow P_5^0\wedge P_5(\mathrm{i})$, *where*

2:6. $P_5(\mathrm{i})$: *In every linearly ordered set L of an inaccessible cardinality there is a disjoint family of cardinality sL of open non empty intervals of L.*

3. Four alternatives on trees.

3:1. Proposition P_{16} — (*Tree alternative*): *Every infinite tree of regular cardinality is equinumerous to an own subchain or to an own sub-antichain.*

Let us consider the following proposition as well.

3:2. P_{17} (*General tree alternative*): *Every infinite tree T contains a chain of cardinality $|T|$ or an antichain of cardinality $cf\,|T|$.*

3:3. $P_{\overline{17}}$ (dual of P_{17}) *Every infinite tree T contains an antichain of cardinality $|T|$ or a chain of cardinality $cf\,|T|$.*

3:4. P_{16}^{∞} (*Unrestricted tree alternative*): *Every infinite tree is equinumerous to an own chain or to an own antichain.*

In KUREPA 1935 p. 130 was considered the following

3:5 Proposition P_2. *Every infinite tree is D-reflexive* (i.e. equinumerous to some own degenerated subtree).

4. Theorem. *The statements P_2, P_{16}, P_{17}, $P_{\overline{17}}$ are pairwise equivalent. The statement P_{16}^{∞} is false.*

Proof. The theorem 4 will be proved in proving the lemma (4:8:1) and the following implications:

$$\begin{array}{lll} 4:1. & P_2 \Leftrightarrow P_{16} & \\ & \Updownarrow \quad \searrow \quad \Uparrow & \\ & P_{17} \quad\; P_{\overline{17}} & \end{array}$$

4:2. Proof of $P_2 \Rightarrow P_{17}$. In virtue of P_2 every infinite tree T contains a D-subtree T_d of cardinality $|T|$. Now,

4:2:1. $T_d = \bigcup_a [a, .)_{T_d}$, $(a \in R_0 T_d)$; T_d being degenerated, the summands in (4:2:1) are disjoint chains and one has

(4:2:2) $|T_d| = \Sigma_a |[a, .)_{T_d}|$, $(a \in R_0 T_d)$, from where we infer that the following alternative holds:

Some summand in (4:2:2) equals $|T_d|$ or

$$\mathrm{cf}|T_d| = |R_0 T_d|.$$

On the other hand, the summands in $(4:2:1)_2$ and the set $R_0 T_d$ being chains and an antichain respectively, this means exactly that P_{17} holds.

4:3. Proof of $P_{17} \Rightarrow P_2$. We have to prove that P_{17} implies that every infinite tree T contains a D-subset T_d of cardinality $|T|$. Let us consider the following possible two cases: $|T|$ is regular, $|T|$ is irregular. If $|T|$ is regular, then the requested implication $P_{17} \Rightarrow P_2$ is obvious. Therefore let us restrict to a singular transfinite tree T. A further reduction is that T is a ramified sequence, i.e. that.

(4:3:1) $\gamma T = \gamma [a]_T$ for every $a \in T$. Moreover, we could restrict us to the case that $\gamma T = \omega_\alpha$, $|T| = |\gamma T|$ (cf. KUREPA, Đ. (1935), pp. 108–9, § 3.). Now, if T has a chain L of cardinality $|T|$, it is sufficient to put $T_d := L$. If every chain of T is $< |T|$, then by hypothesis P_{17} there exists an antichain A of T such that $|A| = cf|T|$. The number ω_α being singular, let τ be the regular ordinal of cardinality $cf|T|$; let then $l_i (i < \tau)$ be a sequence of ordinal numbers such that $|l_i| (i < \tau)$ be a strictly increasing sequence of cardinals, the sum of which equals $|T|$. Let $(a_i)\ (i < \tau)$ be a well-ordering of A; since by assumption (4:3:1) one has

$$\gamma [a]_T = \omega_\alpha (a \in A),$$

it is sufficient to consider a sequence of points b_i such that

$$a_i < b_i \in R_{l_i} (a_i .)_T \text{ and to set}$$

(4:3:2) $T_d = \bigcup_i [a_i, b_i]_T \ (i < \tau)$

in order to be aware that the summands in (4:3:2) are pairwise incomparable chains, and that T_d is a requested D-set of cardinalitiy $|T|$. Q.E.D.

4:4. Proof of $P_2 \Rightarrow P_{16}$. In fact, every D-subset T_d of a T of a regular cardinality $\aleph_\alpha$ is such that $|R_0 T_d| = \aleph_\alpha$ or for some $a \in T_0$ the chain $R_d [a,.)$ has $|T|$ members.

4:5. Proof of $P_{16} \Rightarrow P_2$. Let T be any infinite tree; to prove that P_{16} implies the existence of a D-subset $T_d \subset T$ of cardinality $|T|$. We could assume that $|T|$ is singular and that the conditions (4:3:1), (4:3:2) be satisfied. Let τ be the regular initial number of cardinality $cf|T|$; let e_i, $(i < \tau)$ be any strictly increasing sequence of initial ordinals such that $\sup e_i = \gamma T (i < \tau)$. Put

$$S := \bigcup_i R_{e_i} T (i < \tau).$$

Then S is a ramified sequence of the rank $\gamma S=\tau$. If S contains some antichain A of cardinality $cf|T|$, then by the arguments as in the section 4:3 one obtains a requested D-subset T_d of T. If every antichain of S is $<|\tau|$, then in particular every row of S is $<|\tau|$ and consequently $|S|=|\tau|$; thus $|S|$ is regular; therefore, by assumption P_{16} the tree S contains a chain or an antichain of $|\tau|$ points; the last case being excluded, let then $L_0=a_0, a_1, \ldots, a_i, \ldots (i<\tau)$ be a chain in S of the cardinality $|\tau|$. Then for every $i<\tau$ we have the interval $J_i: T(a_i, a_{i+1})$ of all points of T located between a_i, a_{i+1}; one has $|J_i|=|e_i|$. Then it suffices to denote by T_d the union of all the intervals $J_i (i<\tau)$, in order to be aware that T_d is a chain in T and that $|T_d|=|T|$.

4:6. Proof of $P_2 \Rightarrow P_{\overline{17}}$. As a matter of fact, if T is infinite and T_d of cardinality $|T|$, then the disjoint partition of T_d into chains $T_d[a, .)$, a running through $R_0 T_d$ implies that some of these chains is $\geqslant \mathrm{cf}|T|$ or that $|R_0 T_d|=|T|$.

4:7. Proof of $P_{\overline{17}} \Rightarrow P_{16}$. In order to deduce P_{16} from $P_{\overline{17}}$ we could restrict us to the case that: $\gamma T=\omega_\alpha$, $|T|=\aleph_\alpha=$ regular $|R_i T|$ $(i<\gamma T)$ and that there is no maximal chain in T(cf. Kurepa, Đ. (1935), p. 100), T containing no chain of cardinality $\aleph_\alpha (=|\tau|T|)$, the assumption $P_{\overline{17}}$ says that T contains an antichain A of cardinality $|T|=\aleph_\alpha$, what proves that P_{16} holds. Consequently all implications in (4:1) are proved.

4:8. Remark. In connexion with the statement P_{16}, one should add that the condition that the cardinality of the tree be regular is indispensable.

In fact we have the following

4:8:1. *For every transfinite singular cardinal number n there is a tree $T(n)$ of power n and in which every chain is $<n$ and every antichain is $\leqslant \mathrm{cf}\, n$.*

Proof. Let τ be the initial ordinal number of power $\mathrm{cf}\, n$ and let n_i $(i<\tau)$ be a strictly increasing sequence of cardinal numbers such that $\Sigma_i n_i=n$, $(i<\tau)$. For every $i<\tau$ let L_i be a well-ordered set of cardinality n_i. There is no loss of generality to assume that the sets $L_i (i<\tau)$ are pairwise disjoint; let $T(n)$ be the cardinal sum of the well-ordered sets L_i; then $T(n)$ has the requested property: every chain in $T(n)$ is $<n$ and every antichain in T is $\leqslant|\tau|$.

The lemma 4:8:1 is to be compared to the following

4:8:2. Lemma: *Every transfinite tree T of a singular cardinality n satisfying* $\sup_L\{|L| \mid L$ *is a subchain of* $T\}<|T|$ *contains an antichain of the cardinality* $|T|$. The content of the lemma is a corollary of the theorem 2 p. 109 in Kurepa (1935).

5. Relation between an ordered chain L and its square L^2. Proposition P_{13}.

In 1950 we found an inexpected tie between the celullarities of L, L^2, L^n $(n \in N)$ translating an interesting connexion between subdivisions in L and pavings in L^n. Let us formulate the following.

5:1. Proposition P_{13}. *For every ordered chain L and every disjoint system $\mathcal{F}$ of open sets in the square L^2 the chain L contains a disjoint system of open intervals of the cardinality* $|\mathcal{F}|$ (cf. Đ. Kurepa, 1950, 1952 and in particular 1953 a, 1953 b).

5:2. Theorem. $P_5 \Leftrightarrow P_{13}$ (for the wording of P_5 see § 2:4).

5:3. Proof of $P_5 \Rightarrow P_{13}$. Since $c(L^2) \leqslant s(L^2) = sL$ and, by hypothesis P_5, $cL = sL$, we infer that $c(L^2) \leqslant cL$, thus $c(L^2) = cL$; by P_5 the number cL is reached in L; this proves that P_{13} is holding[1)].

5:4. Proof of $P_{13} \Rightarrow P_5$. Assume $sL = \aleph_\alpha$; we have to prove that, under P_{13}, PL contains a disjoint system of cardinality sL of open intervals of L. Let us consider a complete bipartition T of L into intervals; then $(T, \supset)$ is a tree of nonoverlapping intervals of L; one has necessarily $\gamma T < \omega_{\alpha+1}$. In opposite case, $\gamma T \geqslant \omega_{\alpha+1}$, thus $\gamma T = \omega_{\alpha+1}$ because the relation $\gamma T > \omega_{\alpha+1}$ would imply that there is some $I \in R_{\omega_{\alpha+1}} T$, hence that there exists an $\omega_{\alpha+1}$ — sequence of strongly increasing or decreasing intervals of L, contradicting the assumption that $sL = \aleph_\alpha$. Now the relation $\gamma T = \omega_{\alpha+1}$ is not possible neither, because if we consider for every $X \in T$ the immediate followers X_0, X_1 of X such that $X_0 < X_1$, the corresponding rectangles $X_0 \times X_1$ $(X \in T)$ would yield a system of $\aleph_{\alpha+1}$ disjoint rectangles in L^2 and in virtue of P_{13} a system of $\aleph_{\alpha+1}$ disjoint intervals of L, in contradiction to the assumption that $sL = \aleph_\alpha$.

6. Atomization in L and paving in L^n for any $1 < n \in N$.

There is an interesting link between any atomization (or a complete development of L) and the corresponding paving of L^n (cf. Đ. Kurepa (1950).

6.1. Theorem. *For every $1 < n \in N$ and for every ordered infinite chain L the number $c(L^n)$ is reached and equals $sL (= s(L^n))$. More precisely, if T is any complete bipartition of L, and X_0, X_1 for any $X \in T$ is a bipartition of the interval X into disjoint intervals X_0, X_1 such that $X_0 < X_1$, then $\{int(X_0 \times X_1) \mid X \in T\}$ is a disjoint system of open sets of L^n of the cardinality $c(T^n)$; thus $c(T)$ is reached. Moreover, the set*

6:2. $L^n \setminus \cup_X b(X)$ $(X \in T)$ where

6:3. $b(X) := \{X_{d_0} \times X_{d_1} \times \cdots \times X_{d_{n-1}};\ (d_0, d_1, \ldots, d_{n-1}) \in \{0, 1\}^{I_n}$

is a subset of the diagonal $\Delta := \{(l)_n l \in L$ of the cube L^n; here $I_n := \{0, 1, \ldots, n-1\}$, $(l)_n := \underbrace{(l, l, \ldots, l)}_{n}$. If L is continuous every closed brick $\overline{(X, d)}$ has the unique point $(\sup X_0)_n$ in common with the diagonal

6:4. $\Delta := \{(x, x, \ldots, x) : x \in L\}$ of the cube L^n (cf. § 5 in Đ. Kurepa (1950) p. 113).

Proof. Given $1 < n \in N$; for every dyadic n-sequence $d = (d_0, d_1, \ldots, d_{n-1})$ (thus $d \in \{0, 1\}^{I_n}$) let us consider the corresponding brick

6:5. $(X, d) := X_{d_0} \times X_{d_1} \times \cdots \times X_{d_{n-1}}$. Then we have the following lemma.

---

1) Let us remark that for every L the number $c(L^2)$ is reached; the same holds for $c(L^n)$ for every $1 < n \in N$; probably, it is an independent statement that cL is reached as well (*the number cL is reached for every L such that cL is not inaccessible*; cf Đ. Kurepa 1935 p. 110, in particular Theor. 3. Let us remark that $c(L^2)$ is reached for every chain L (see Đ. Kurepa (1975) theor. 1—5; cf. also § 5:6, 5:6:6. of the present paper).

6.6. Lemma. *For every* $X \in T$, $X = \bigcup_d X(d)$ $(d \in \{0, 1\}^{I_n}, I_n = \{0, 1, \ldots, n-1\})$ *is a disjoint partition of the cube* X^n.

First, if d, e are two distinct dyadic n-sequences, then $d_i \neq e_i$ for some $i \in I_n$; thus $X_{d_i} \neq X_{e_i}$, hence X_{d_i}, X_{e_i} are disjoint and therefore are so also the corresponding bricks $X(d)$, $X(e)$. On the other side, let $x: = (x_0, x_1, \ldots, x_{n-1}) \in X^n$; then $i \in I_n$ implies that $x_i \in X_{d_i}$ either for $d_i = 0$ or for $d_i = 1$; hence the sequence $d: = (d_0, d_1, \ldots, d_{i-1})$ is uniquely determined by the member $x \in X^n$ and one has $x \in X^n$ (d).

6:7. Lemma. *Let* $\Delta: = \{(x, x, \ldots, x): x \in L\}$ *be the diagonal of the cube* L^n; *then for every complete bipartition* $(T, \supset)$ *of* L *and every interval* $X \in T$ *we have the pile* $b(X)$ *of all* $2^n - 2$ *bricks of the form* 6:5 *where* d *is neither the constant* n*-sequence* 0 *nor the constant* n*-sequence* 1.

The union

6:8. $\bigcup(Y \setminus \Delta)$, $(Y \in b(X), X \in T)$ is a disjoint paving of the diagonalless cube $L^n \setminus \Delta$. This paving is such that every brick has interior points and is a maximum paving both in the sense that the cardinality of the system of all bricks equals $c(L^n)$ and further that the system of bricks is not further extensionable.

At first, for every brick $X(d)$ and every interior point $x_i \in X_{d_i}$ the point $(x_0, x_1, \ldots, x_{n-1})$ is an interior point of the cube X^n. Further, for the system B of all bricks we have $|B| = |T|\,(2^n - 2) = |T|$, because T is infinite. Thus

6:9. $c(L^n) \geqslant |B| = |T|$. On the other side

6:10. $|T| \geqslant sL$ because if f associates to every $X \in T$ an interior point $f(X)$, then $\{f(X): X \in T\}$ is everywhere dense in L^n and is of a cardinal number $\leqslant |T|$. From 6:9, 6:10 we infer that $c(L^n) \geqslant sL$, and consequently $c(L^n) = sL$, because obviously, $c(L^n) \leqslant s(L^n)$.

Finally, if L is continuous, then for every $X \in T$ and every $(X, d) \in b(X)$, the closed brick $\overline{(X, d)}$ equals $\overline{X}_{d_0} \times \overline{X}_{d_1} \times \cdots \times \overline{X}_{d_{n-1}}$ and contains the point and only the point $(\sup X_0)_n$ of Δ because $\{\sup X_0\} = \overline{X}_0 \cap \overline{X}_1$.

The theorem 6:1 is completely proved.

6:11. If instead of bipartitions we consider m-partitions for any $1 < m \in N$, then for any complete m-partition T of L, for any $X \in T$, we have a disjoint m-partition $X = X_0 \cup X_1 \cup \cdots \cup X_{m-1}$ of X such that $X_0 < X_1 < \cdots < X_{m-1}$. For any non constant $d \in I_m^{I_n}$, we would consider the corresponding brick (X, d) and the corresponding pile $b(X)$ of all $m^n - m$ bricks of the form 6:5; again, in the system $B: = \bigcup_X b(X)$ $(X \in T)$ of all bricks, the interiors int $B: = \{\text{int } Y: : Y \in B\}$ constitute a disjoint system S of non empty open sets in L^n such that $|S| = c(L^n) = s(L^n) = sL$. Thus, in particular, $c(L^n)$ is reached.

6:12. Remark. Already $\{\text{int}(X_0 \times X_1 \times \cdots \times X_{n-1}): X \in T\}$ is a maximum disjoint system in L^n and has $|T|$ i.e. sL members (because $|T| = sL$).

6:13. Remark. T being a complete m-development of a continuous L the system (5:6:7) of all bricks is a complete paving of $L^n \setminus \Delta L^n$; a brick $X(d)$ satisfies $X(d) \cap \Delta \neq \varnothing$ if and only if the set $\{d_0, d_1, \ldots, d_{n-1}\}$ is formed of two consecutive members of I_n. This follows immediately from the equivalence: $\overline{X}_i \cap \overline{X}_j \neq \varnothing \Leftrightarrow |i - j| \leqslant 1$.

7. Propositions P_2 (i), P'_{14}, P_{14}, P'_{15}, P_{15}.

We have the following consequence P_2 (i) of P_2.

7:1. :P_2 (i): *Every tree of an inaccessible cardinality is D-reflexive.*

7:2. *Proposition P'_{14} (chain $\times$ antichain hypothesis on trees)*: Every tree T satisfies

7:3. $|T| \leqslant k_c T.\ k_a T$, where

7:4. $k_c T :=\sup_L |L|$, L running through the system of all chains of T.

7:5. $k_a T :=\sup_A |A|$, A running through the system of all antichains in T.

7:6. Proposition P'_{15}. *Every tree T is the union of $k_a T$ subchains of T* (see Kurepa Đ. (1963) p. 30, Th. 3.3).

7:7. Proposition $P_{14} := P'_{14} \wedge P_2$ (i).

7:8. Proposition $P_{15} : P_{15} \wedge P'_2$ (i).

7:9. Theorem $P'_{14} \Leftrightarrow P'_{15}$ (= Theorem 3.3 p.30 in Kurepa, Đ. (1963)).

7:10. Theorem: $P_{14} \Leftrightarrow P_{15}$. *The propositions P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_{17}, $P_{\overline{17}}$ are mutually equivalent* (see Kurepa, Đ. 1935 p. 130—2 for definitions and equivalences of P_1, P_2, P_3, P_4, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}).

8. A consequence of the hypotheses P_i ($i=1, 2, \ldots ,$). Maximum antichain tree hypothesis [MATH].

The question is about the following.

8.1. Maximum antichain tree hypothesis [MATH]: *Every tree contains a maximum antichain*; in other words if for any tree T we define

8:2. $\check{s}T :=\sup_X |X|$ ($X \subset T$, X is antichain), then T contains an antichain, say T_A, of cardinality $\check{s}T$.

8:3. Theorem: *Every tree T such that $|T| > k_c T$ contains a maximum antichain.*

For every ordered set E we put

8:3:1. $k_c E :=\sup_L |L|$, L being a chain in E.

8:3:2. Proof of the theorem 8:3. If T contains a D-subset T_d such that $|T_d| = |T|$, then $|R_0 T_d| = |T_d| = |T|$; thus $R_0(T_d)$ is a maximum antichain of T. Therefore let us assume that T be non D-reflexive. Then T contains a distinguished ramified sequence S of power cf$|T|$ (Kurepa 1935 th. 2, p. 109). The height γS is accessible-otherwise the condition $|T| > k_c T$ should be violated. Thus $\gamma S = \omega_{\beta+1}$ and every maximum antichain should have either $\aleph_\beta$ elements or $\aleph_{\beta+1}$ elements. In both cases it does exist; in the first case, every node has $\aleph_\beta$ points, in the second case the maximum antichain exists because the number $\aleph_{\beta+1}$ as the supremum of the cardinalities of antichains in T, one antichain has necessarily $\aleph_{\beta+1}$ members.

Thus the theorem 8:3 is proved completely.

8:4 Theorem: *Every tree T of a cardinality that is not inaccessible contains a maximum antichain $T_A^{\cdot}$.*

Proof of 8:4.: In virtue of 8:3 we restrict us to the case $|T| = k_c T$.

8:4:1. Let $T_0 = \{x \mid x \in T, \check{s}[x, .)_T < \check{s}T\}$.
We have either $\check{s}T_0 = \check{s}T$ or $\check{s}T_0 < \check{s}T$.

8:4:2. Case $\check{s}T_0 = \check{s}T$. We have $T_0 = \cup T_0[x, .)$ $(x \in R_0 T_0)$ and obviously $k_c[x, .)_{T_0} \leqslant k_c[x, .)_T < \check{s}T$; therefore $|R_0 T_0| \geqslant \mathrm{cf}\, \check{s}T$. If $|R_0 T_0| = \check{s}T_0$, then it is sufficient to set $T_A = R_0 T_0$. If $|R_0 T_0| < \check{s}T_0$, then there is a subset $A \subset R_0 T_0$ of cardinality $\mathrm{cf}\, T_0$ such that the set

$$B := \cup T_0[a, .), (a \in A)$$

satisfies $\check{s}B = \check{s}T$; in particular, for every τ-sequence k_i of cardinals $\uparrow \check{s}T$, one has a τ-sequence $a_i \in R_0 B$ such that $\check{s}[a_i, .) \geqslant k_i$; thus there is an antichain $A_i \subset T_0[a_i, .)$ of cardinality $\geqslant k_i$;, then it is sufficient to set $T_A^{\cdot} := \cup A_i (i < \tau)$.

8:4:3. Case $\check{s}T_0 < \check{s}T$; set $T_1 := T \setminus T_0$; then $|T_1| = |T|$ and

8:4:4. $\check{s}[x, .)_{T_1} = \check{s}T_1 = \check{s}(T) := k$ for every $x \in T_1$.

8:4:4:1. First subcase: The number $\check{s}T$ is singular; thus there exists a τ-sequence of antichains $A_i \subset T_1$ such that $\sup_{i<\tau} |A_i| = \check{s}T$; in particular, there is an antichain $B := \{b_i\}_{i<\tau}$ in T_1 of cardinality $|\tau|$. Since $\check{s}[b_i, .)_{T_1} = \check{s}T$ for every $i < \tau$, let us consider any antichain $B_i \subset [b_i, .)_{T_1}$ such that $|B_i| \geqslant k_i$; then it is sufficient to set $C = \cup_{i<\tau} B_i$ in order to get an antichain of cardinality $\check{s}T$.

8:4:4:2. Second subcase: $\check{s}T$ is regular, thus $\check{s}T$ is inaccessible (because we assumed $\check{s}T$ to be non isolated). Of course, there is no maximum chain in T, because if there were one, say L, then by assumption $|T| = k_c T$ one should have $|L| = k_c T = |T|$; since $\check{s}[a, .) = k_c T$ for every $a \in T_1$, there is in particular for every $l \in L$ a point l^+ such that $l < l^+ \in T_1 \setminus L$; then the set L^+ of all points is an antichain of cardinality $|L|$, thus L^+ would be a maximum antichain in T. Thus the theorem 8:4 is completely proved.

8:4:5. Corollary. *Every tree of a singular cardinality contains a maximum antichain.*

8:5. Theorem. *The maximum antichain tree hypothesis is equivalent to the following*

8:6. L_1(i) – Hypothesis. *Every totally ordered space of an inaccessible separability contains a maximum disjoint system of open sets.*

8:7. Proof of the implication MATH$\Rightarrow L$(i).

8:7:1. In Kurepa 1935 p. 111 was proved that for every decreasing tree $(T, \supset)$ of a non inaccessible cardinality the system $T^d := \{A \setminus B \mid A, B \in T\}$ contains a maximum disjoint subsystem.

8:7:2. Let L be any ordered chain of an inaccessible c-number cL. Let $(T, \supset)$ be a complete bipartition (atomization) of L; then $|T| = cL$ (see Kurepa 1935 p. 120); thus T is a tree of the inaccessible cardinality cL; in virtue of MATH T contains a maximum antichains T_A i.e. a disjoint system of intervals of cardinality $\check{s}T$; it is easy to see that $|T_A| = |T| = cL$, thus L contains a maximum disjoint system of intervals.

8:8. Proof of L(i) $\Rightarrow$ MATH.

8:8:1. Let T be any tree; if $|T| > k_c T$ or if $|T|$ is not inaccessible, then T contains a maximum antichain (see theorem 8:4.). Therefore let us assume that $|T|$ is inaccessible non countable $\aleph_\sigma$ and that every chain in T be $<|T|$; then (see KUREPA Đ. (1935) p. 109, Théorème 2) T contains a distinguished ramified sequence S of rank $\gamma T = \omega_\sigma$, in particular, every node of S is infinite[1])

8:8:2. Further, for every node N of ST let $(N, \leqslant_N)$ be a total order of N; then the "*natural ordering*" $(T, \prec)$ of $(T, \leqslant)$ is the following one: for $a, b \in T$ $a \prec b :\Leftrightarrow a \leqslant b \vee (a \| b \wedge a' \leqslant_N b')$, $a' \leqslant a$, $b' \leqslant b$ and where N is the node of $(T, \leqslant)$ satisfying $n \in N \Rightarrow (., n) = (., a)_T \cap (., b)_T$; in other words, a', b' are the distinct points such that $a' \leqslant a$, $b' \leqslant b$ and $T(., a')\, T(., b')$.

8:8:3. One proves readily that $(T, \prec)$ is a total order extending $\leqslant$ and $\leqslant_N$ for every node N of T. Moreover the orders $(T, \leqslant)$, $(T, \prec)$ have the following two properties (cf. KUREPA, Đ. (1935), 128–129).

8:8:4. *For every point $t \in T$ the set $S := (t, .)_{(T, \leqslant)}$ is a portion of the chain $(T, \prec)$; in other words, if the set S contains two points a, b then S contains the whole interval $[a, b]_{(T, \prec)}$.*

8:8:5. Lemma. *Every interval $[a, b)_{(T, \prec)}$, where $a \prec b$, contains a non empty portion* (cf. KUREPA 1935 p. 128–129) *and some $t \in T$ satisfies*

8:8:6. $[t, .)_{(T, \leqslant)} \setminus [b, .)_{(T, \leqq)} \subset (a, b)_{(T, \prec)}$

If then S is any maximum disjoint system of intervals in $(T, \prec)$, and if for every interval $I \in S$ we denote by $t = t(I)$ a point of T satisfying (8:8:6) for $a = \inf I$, $b = \sup I$, then $\{t(I) \mid I \in S\}$ is a maximum antichain in $(T, \leqslant)$. The proposition L(i) $\Rightarrow$ MATH is completely proved.

9. Problem. Is there a model of Set Theory in which the statements P' (v. 1:2:1), TA (v. 3:1) are equivalent?

(cf. also KUREPA, Đ. [1968 a], [1968 b], 1974; in particular, in [1974 a] relevant bibliography is indicated).

BIBLIOGRAPHY:

Easton, W. B.: 1970: *Powers of regular cardinals*, Annals of Math. Logic 1 (1970), 139—178.

Kurepa, Đuro: 1935: *Ensembles ordonnés et ramifiés*. Thèse, Paris, (1935) VI + 138 + + II; Publ. math. Belgrade, 4 (1935) 1—138.

Kurepa Đ., 1936: *L'hypothèse de ramification*, Comptes rendus 202, 1936, 185—187.

Kurepa Đ., 1937: *Transformations monotones des ensembles partiellement ordonnés*, Comptes rendus, 205, 1937, 1033—1035.

Kurepa Đ., 1940—41: *Transformations monotones des ensembles partiellement ordonnés*, Revista de Ciencias, Lima, No. 434, 42, 1940, 827—846; No. 437, 43, 1941, 483—500.

Kurepa Đ., 1950: *La condition de Suslin et une propriété caractéristique des nombres réels*, Comptes rendus, Paris, 231, 1950, 1113—1114.

---

[1]) A node of T is every maximal subset N of T such that $T(., x) = T(., y)$ for every $\{x, y\} \subset N$ (cf. Kurepa 1935 p. 72, ...). In other words, if $x \sim y :\Leftrightarrow T(., x(= T(., y)$, then $\sim$ is an equivalence relation in $(T, \leqslant)$; the classes $T/\sim$ are just nodes of $T, \leqslant)$.

Kurepa, Đuro: 1952: *Sur une propriété caractéristique du continu linéaire et le problème de Suslin*, Publ. Math. Inst. Ac. serbe Sci. Belgrade, 4, 1952, 97—108.

Kurepa Đ., 1953 a: *Sur une hypothèse de la théorie des ensembles*, Comptes rendus, Paris, 256, 1953, 564—565, (séance du 09. 02. 1953.)

Kurepa Đ., 1953 b: *Sur un principe de la théorie des espaces abstraits*, Ibidem 655—657; séance du 16. 02. 1953.

Kurepa Đ., 1958—9: *Sull ipotesi del continuo*, Rendiconti del Seminario Matematico, Torino, 18, 1958—59, 11—20.

Kurepa, Đ., 1959: *Sur une proposition de la théorie des ensembles*, *C.* R. Sci, Paris 249, 1959, 2698—99.

Kurepa Đ., 1963: *Star number and antistar number of ordered sets and graphs*. Glasnik Mat. Fiz. Astr. Zagreb, 18 (1963) 27—37.

Kurepa Đ., 1968 a: *On several continuum hypotheses*; Proc. international summer school, Varenna 1968, 57—64 (Aspects of Mathematical Logic, Roma 1969, 1—285).

Kurepa Đ., 1968 b: *Around the general Suslin problem*. Proc. Intern. Symp. on Topology in Herceg Novi 1968, Beograd 1969.

Kurepa Đ., 1974: *On trees or ramifications*, Mathematica Balkanica, Beograd 4 (1974) 371—382.

Ovaj rad je financirala Zajednica za nauku SR Srbije preko Matematičkog instituta, Beograd.

Address:
KUREPA, ĐURO
ZAGREBAČKA, 7
Beograd, Yugoslavia