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AN APPROACH TO FIXED POINTS IN PRODUCT SPACES

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Abstract. The paper deals with a class of mappings defined on products of metric spaces satisfying some conditions with respect to the metric defined on the product space. As a consequence it is shown that such a mapping possesses unique fixed point under suitable restrictions.

Introduction.

A well-known result due to Banach is that: A self-mapping f defined on a complete metric space (X, d) satisfying

$$d(f(x), f(y)) \leqslant k d(x, y) \tag{1}$$

for some k, $0<k<1$; $x, y \in X$, has a unique fixed point in X. Singh [13] established the same theory for a mapping f which satisfies

$$d(f(x), f(y)) \leqslant \alpha\, d(x, f(x)) + \beta\, d(y, f(y)), \tag{2}$$

for all $x, y \in X$ and $0<\alpha+\beta<1$ instead of (1), which is a modification of the condition taken by Kannan [3].

In a recent paper [1] Dass and author have considered a class of mappings defined on products of metric spaces which satisfy Lipschitz condition in each variable separately (abbreviated LCIEVS) viz.

$$\left.\begin{aligned} d^{(p)}(f(x_1, z_2), f(y_1, z_2)) \leqslant k d^{(p)}((x_1, z_2), (y_1, z_2)) \\ d^{(p)}(f(z_1, x_2), f(z_1, y_2)) \leqslant k d^{(p)}((z_1, x_2), (z_1, y_2)) \end{aligned}\right\} \tag{A}$$

where $x_1, y_1, z_1 \in X_1$ and $x_2, y_2, z_2 \in X_2$, and have studied as to when these satisfy Lipschitz condition jointly and hence contraction or contractive. This study helped in to extend almost all results regarding fixed point theorems in single spaces whose basis is contraction mapping (i.e. (1)) to products of metric spaces for suitable choices of constants.

In this correspondence we consider a class of mappings defined in products of metric spaces as follows:

Let $X=X_1\times X_2$ be the cartesian product of two metric spaces (X_1, d_1) and (X_2, d_2) with the product metric $d^{(p)}$ defined as

$$d^{(p)}((x_1, x_2), (y_1, y_2))=\begin{cases}[d_1(x_1, y_1)]^p+[d_2(x_2, y_2)]^p & 0<p\leqslant 1\\ [[d_1(x_1, y_1)]^p+[d_2(x_2, y_2]^p]^{1/p} & 1<p\leqslant\infty\\ \max[d_1(x_1, y_1), d_2(x_2, y_2) & p=\infty\end{cases}\tag{B}$$

for $(x_1, x_2), (y_1, y_2)\in X$.

Let f_1 and f_2 be two self mappings defined on X_1 and X_2 respectively and $f=f_1\times f_2$ be the mapping defined on X into itself satisfying: for any $x_1, y_1, z_1\in X_1, x_2, y_2, z_2\in X_2$,

$$\left.\begin{aligned}d^{(p)}(f(x_1, z_2), f(y_1, z_2))\leqslant\alpha\, d^{(p)}((x_1, z_2), f(x_1, z_2))+\beta\, d^{(p)}((y_1, z_2), f(y_1, z_2))\\ d^{(p)}(f(z_1, x_2), f(z_1, y_2))\leqslant\alpha\, d^{(p)}((z_1, x_2), f(z_1, x_2))+\beta\, d^{(p)}((z_1, y_2), f(z_1, y_2))\end{aligned}\right\}\tag{C}$$

where α, β are any positive real numbers.

We shall investigate as to when these conditions imply the following:

$$d^{(p)}(f(x_1, x_2), f(y_1, y_2))\leqslant\alpha'\, d^{(p)}((x_1, x_2), f(x_1, x_2))+\beta'\, d^{(p)}((y_1, y_2), f(y_1, y_2))\tag{D}$$

where α' and β' are constants depending upon α and β.

Further, we seek for the values of α, β for which $\alpha'+\beta'<1$.

This observation is important in the sense that almost all results to date regarding fixed point theorems in single spaces whose basis is (2) (refer [2—9, 13, 14] etc.) can be extended to product spaces for the said type of functions.

2. We prove the following:

Theorem 1. *Let $f_1: X_1\to X_1$ and $f_2: X_2\to X_2$ be any mappings. Then a mapping $f=f_1\times f_2$ defined on $X=X_1\times X_2$ into itself satisfying* (C) *for some α, β where the metric $d^{(p)}$ defined on X is as in* (B) *would satisfy condition* (D) *where*

$$\alpha'+\beta'=\begin{cases}3(\alpha+\beta), & 0<p\leqslant 1\\ 2^{1-1/p}\,3(\alpha+\beta) & 1<p<\infty\\ 2\quad 3(\alpha+\beta) & p=\infty\end{cases}$$

Proof. We have

$$\begin{aligned}(3)\quad & d^{(p)}(f(x_1, x_2), f(y_1, y_2))\leqslant d^{(p)}f(x_1, x_2), f(x_1, y_2))+d^{(p)}(f(x_1, y_2), f(y_1, y_2))\\ &\leqslant\alpha\, d^{(p)}((x_1, x_2), f(x_1, x_2))+\beta\, d^{(p)}((x_1, y_2), f(x_1, y_2))+\\ &\quad\alpha\, d^{(p)}((x_1, y_2), f(x_1, y_2))+\beta\, d^{(p)}((y_1, y_2), f(y_1, y_2))\\ &=\alpha\, d^{(p)}((x_1, x_2), f(x_1, x_2))+(\alpha+\beta)\, d^{(p)}((x_1, y_2), f(x_1, y_2))\\ &\quad+\beta\, d^{(p)}((y_1, y_2), f(y_1, y_2)).\end{aligned}$$

Case 1. When $0<p\leqslant 1$, then

R. H. S of (3) $=\alpha[[d_1(x_1, f_1(x_1))]^p+[d_2(x_2, f_2(x_2)]^p]+(\alpha+\beta)[[d_1(x_1, f_1(x_1))]^p+[d_2(y_2, f_2(y_2))]^p]+\beta[[d_1(y_1, f_1(y_1))]^p+[d_2(y_2, f_2(y_2))]^p]$

$=(2\alpha+\beta)[d_1(x_1, f_1(x_1))]^p+\alpha[d_2(x_2, f_2(x_2))]^p+\beta[d_1(y_1, f_1(y_1))]^p$

(4) $+(\alpha+2\beta)[d_2(y_2, f_2(y_2))]^p$

(5) $\leqslant(2\alpha+\beta)d^{(p)}((x_1, y_1), f(x_1, y_1))+(\alpha+2\beta)d^{(p)}((x_2, y_2), f(x_2, y_2))$

Case 2. When $1<p<\infty$, then

R. H. S of (3) $=\alpha[[d_1(x_1, f_1(x_1))]^p+[d_2(x_2, f_2(x_2))]^p]^{1/p}+(\alpha+\beta)[[d_1(x_1, f_1))]^p+[d_2(x_2, f_2(x_2))]^p]^{1/p}+\beta[[d_1(y_1, f_1(y_1))]^p+[d_2(y_2, f_2(y))]^p]^{1/p}$

$\leqslant\alpha[d_1(x_1, f_1(x_1))+d_2(x_2, f_2(x_2))]+(\alpha+\beta)[d_1(x_1, f_1(x_1))$

(6) $+d_2(y_2, f(y_2))]+\beta[d_1(y_1, f_1(y_1))+d_2(y_2, f_2(y_2))]$.

$\leqslant(2\alpha+\beta)[d_1(x_1, f_1(x_1))+d_2(x_2, f_2(x_2))]$

(7) $+(\alpha+2\beta)[d_1(y_1, f_1(y_1))+d_2(y_2, f_2(y_2))]$

$\leqslant(2\alpha+\beta)\delta d^{(p)}((x_1, x_2), f(x_1, x_2))+(\alpha+2\beta)\delta d^{(p)}((y_1, y_2), f(y_1, y_2))$

where $\delta=\sup\left\{\dfrac{d_1(z_1, f_1(z_1))+d_2(z_2, f_2(z_2))}{[d_1(z_1, f_1(z_1))]^{(p)}+[d_2(z_2, f_2(z_2)^p]^{1/p}}\Big/ (z_1, z_2)\neq f(z_1, z_2)\in X.\right.$

It can be easily seen that $\delta=2^{1-1/p}$

Therefore in this case R. H. S. of (3) $\leqslant 2^{1-1/p}[(2\alpha+\beta)d^{(p)}((x_1, x_2), f(x_1, x_2))$

(8) $+(\alpha+2\beta)d^{(p)}((y_1, y_2), f(y_1, y_2))]$.

Case 3. Lastly, for $p=\infty$

R. H. S of (3) $=\alpha\max[d_1(x_1, f_1(x_1)), d_2(x_2, f_2(x_2))]+(\alpha+\beta)\max[d_1(x_1, f_1(x_1)), d_2(y_2, f_2(y_2))]+\beta\max[d_1(y_1, f_1(y_1)), d_2(y_2, f_2(y_2)]$

$\leqslant\alpha[d_1(x_1, f_1(x_1))+d_2(x_2, f_2(x_2))]+(\alpha+\beta)[d_1(x_1, f_1(x_1)$

(9) $+d_2(x_2, f_2(y_2)]+\beta[d_1(y_1, (y_1))+d_2(y_2, f_2(y_2))]$

$\leqslant(2\alpha+\beta)[d_1(x_1, f_1(x_1))+d_2(x_2, f_2(x_2))]$

(10) $+(\alpha+2\beta)[d_1(y_1, f_1(y_1))+d_2(y_2, f_2(y_2))]$

$\leqslant(2\alpha+\beta)2d^{(p)}((x_1, x_2), f(x_1, x_2))+$

(11) $(\alpha+2\beta)2d^{(p)}((y_1, y_2), f(y_1, y_2))$

Thus $d^{(p)}(f(x_1, x_2), f(y_1, y_2))\leqslant\alpha' d^{(p)}((x_1, x_2), f(x_1, x_2))+\beta' d^{(p)}((y_1, y_2), f(y_1, y_2))$

$$(12)\qquad \text{where } \alpha'+\beta'=\begin{cases} 3(\alpha+\beta), & \text{when } 0<p\leqslant 1 \quad \text{(from (5))}\\ 2^{1-1/p}\,3(\alpha+\beta), & \text{when } 1<p<\infty \quad \text{(from (8))}\\ 2\quad 3(\alpha+\beta), & \text{when } p=\infty \quad \text{(from (11))}\end{cases}$$

This completes the proof of the theorem.

Remark. The study made in Theorem 1 can be extended to any finite product as follows:

Let $(X, d^{(p)})$ be the cartesian product of metric spaces $\{(X_i, d_i), i=1, 2, \ldots, n\}$ with

$$d^{(p)}(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \begin{cases} \sum_{i=1}^{n} [d_i(x_i, y_i)]^p, & 0<p\leqslant 1\\ [\sum_{i=1}^{n} [d_i(x_i, y_i)]^p]^{1/p}, & 1<p<\infty\\ \max\,[d_i(x_i, y_i), i=1, 2, \ldots n], & p=\infty. \end{cases}$$

Then, a self mapping f defined on X such that $f=f_1\times f_2\times\cdots\times f_n$ (each f_i is a self-mapping defined on X_i) satisfying

$$d^{(p)}(f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n), f(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n))$$
$$\leqslant \alpha\, d^{(p)}((x_1, x_2, \ldots, x_i, x_{i+1}, \ldots x_n), f(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n))$$
$$+\beta\, d^{(p)}((x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n),$$
$$f(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n))$$

reduces to

$$d^{(p)}(f(x_1, x_2, \ldots, x_n), f(y_1, y_2, \ldots, y_n)) \leqslant \alpha'\, d^{(p)}((x_1, x_2, \ldots, x_n), f(x_1, x_2, \ldots x_n))$$
$$+\beta'\, d^{(p)}((y_1, y_2, \ldots, y_n), f(y_1, y_2, \ldots, y_n))$$

$$\text{where } \alpha'+\beta'=\begin{cases} (2n-1)(\alpha+\beta), & 0<p\leqslant 1\\ (n^{-1/p})(2n-1)(\alpha+\beta), & 1<p<\infty.\\ \text{n.}\ (2n-1)(\alpha+\beta), & p=\infty \end{cases}$$

3. Application to Fixed Point Theorems

To the best of our knowledge, the theory of fixed points whose basis is (2) has not found any attention for products of metric spaces. However, by taking $\alpha'+\beta'<1$ in (D) we get condition (2) and therefore we are in a position to extend the results of [2—9, 13, 14] and others whose basis is (2) to products of metric spaces for the class of mappings under consideration. To fully illustrate our contention we state a result.

Theorem 2. *Let $f=f_1\times f_2$ be a mapping defined on a metric space $(X_1\times X_2, d^{(p)})$ into itself satisfying* (B), *where f_1 and f_2 are self-mappings defined on X_1 and X_2 respectively,*

$$\alpha+\beta<\begin{cases} 1/3 & , \quad 0<p\leqslant 1 \\ 1/3\cdot 2^{1/1p} & , \quad 1<p<\infty \\ 1/3\cdot 2 & , \quad p=\infty \end{cases}$$

and there exists $(x_0, y_0)\in X_1\times X_2:\{f^n(x_0, y_0)\}\ \{f^{n_i}(x_0, y_0)\}\to(x^*, y^*)$.

Then (x^*, y^*) is unique fixed point of f.

We now discuss an example of a function defined on the product of two spaces satisfying (C) *but not* (A).

Example. Let $f=f_1\times f_2:[0, 1]\times[0, 1]\to[0, 1]\times[0, 1]$ be defined as $f(x, y)=(f_1(x), f_2(y))$

$$\text{where } f_1(x) \text{ (or } f_2(y))=\begin{cases} 0 \text{ if } x\,(\text{or } y)\in[0, 1/2) \\ \dfrac{x\,(\text{or } y)}{20} \text{ if } x\,(\text{or } y)\in[1/2, 1] \end{cases}$$

Since f is not continuous, therefore f does not satisfy LCIEVS.

However, for $\alpha=\beta=3/48$, we see that (A) is satisfied.

Remark. The restriction imposed in (12) on α' and β' with $\alpha'+\beta'<1$ are sufficient for the existence of a unique fixed point of f. However, in view of the conclusions in (5), (7) and (10) from (4), (6) and (9) respectively, we can say that these restrictions may not be exhaustive.

The conditions for the mapping $f=f_1\times f_2$ satisfying (A) and (C) can be unified as follows:

$$\begin{aligned} d^{(p)}(f(x_1, z_2))f(y_1, z_2))\leqslant & \,\alpha\, d^{(p)}((x_1, z_2), f(x_1, z_2)) \\ & +\beta\, d^{(p)}((y_1, z_2), f(y_1, z_2)) \\ & +\gamma\, d^{(p)}((x_1, z_2), (y_1, z_2)) \end{aligned}$$

$$\begin{aligned} \text{and } d^{(p)}(f(z_1, x_2), f(z_1, y_2))\leqslant & \,\alpha\, d^{(p)}((z_1, x_2), f(z_1, x_2)) \\ & +\beta\, d^{(p)}((z_1, y_2), f(z_1, y_2)) \\ & +\gamma d^{(p)}((z_1, x_2), (z_1, y_2)). \end{aligned}$$

On the similar lines, it can be shown that

$$\begin{aligned} d^{(p)}(f(x_1, x_2), f(y_1, y_2))\leqslant & \,\alpha'\, d^{(p)}((x_1, x_2), f(x, x_2)) \\ & +\beta'\, d^{(p)}((y_1, y_2), f(y_1, y_2)) \\ & +\gamma'\, d^{(p)}((x_1, x_2), (y_1, y_2)), \end{aligned}$$

$$\text{where } \quad \alpha'+\beta'+\gamma'=\begin{cases} 3(\alpha+\beta)+\gamma, & 0<p\leqslant 1 \\ 2^{1-1/p}\{3(\alpha+\beta)+\gamma\}, & 1<p<\infty \\ 2 \quad 3(\alpha+\beta)+\gamma, & p=\infty. \end{cases}$$

Through this observation, we can extend results due to Reich [10—12] and others to product of metric spaces for suitable choices of α, β, γ.

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