PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 21 (35), 1977, pp. 89—91

A FIXED POINT THEOREM WITH A FUNCTIONAL INEQUALITY

S. A. Husain and V. M. Sehgal

(Received March 18. 1976)

In a recent paper [1], Ćirić investigated mappings f on a complete metric space (X, d) that satisfy the following condition: there exists a constant $k<1$ such that for all $x, y \in X$,

$$(1) \qquad d(fx, fy) \leqslant k \max \{d(x, fx), d(y fy), d(x, y), d(x, fy), d(y, fx)\},$$

and showed that such mappings have a unique fixed point in X. The purpose of this paper is to strengthen Ćirić's result by considering mappings that satisfy a functional inequality.

Throughout this paper, let (X, d) be a complete metric space, R^+ the nonnegative reals and $\varphi:(R^+)^5 \to R^+$ is a continuous function which is non-decreasing in each coordinate variable and satisfies the condition $\varphi(t, t, t, t, t) < t$ for any $t>0$.

The following is the main result of this paper.

Theorem 1. *Let $f: X \to X$ satisfy the condition: for all $x, y \in X$,*

$$(2) \qquad d(fx, fy) \leqslant \varphi(d(x, fx), d(y, fy) d(x, y), d(x, fy), d(y, fx)).$$

If for some $x_0 \in X$

$$(3) \qquad \sup \{d(x_0, f^n x_0): n \in I \text{ (positive integers)}\} < \infty.$$

Then f has a unique fixed point in X.

Proof. For each $n \in I$, let

$$\delta_n = \sup \{d(f^p x_0, f^q x_0): p, q \geqslant n\}$$

Then by (3) $\delta_n < \infty$. Since $\delta_n (n \geqslant 1)$ is a nonincreasing sequence in R^+, there is a $\delta \geqslant 0$ such that $\delta_n \to \delta$. We claim that $\delta = 0$. If $\delta > 0$ then for any $p, q \in I$,

$$d(f^p x_0, f^q x_0) \leqslant \varphi(d(f^{p-1} x_0, f^p x_0), d(f^{q-1} x_0 f^q x_0), d(f^{p-1} x_0, f^{q-1} x_0),$$
$$d(f^{p-1} x_0, f^q x_0), d(f^{q-1} x_0, f^p x_0))$$

Therefore, if $p, q \geqslant n$, it follows that

$$\delta_n \leqslant \varphi(\delta_{n-1}, \delta_{n-1}, \delta_{n-1}, \delta_{n-1}, \delta_{n-1})$$

and hence by the continuity of φ, $\delta \leqslant \varphi(\delta, \delta, \delta, \delta, \delta) < \delta$, a contradiction. Thus $\delta = 0$. This, implies that $\{f^n x_0\}$ is a Cauchy sequence in X and hence, by completeness, there is a $u \in X$ such that $f^n x \to u$. Now, since

$$d(fu, f^{n+1} x_0) \leqslant \varphi(d(u, fu),\ d(f^n x_0, f^{n+1} x_0),\ d(u, f^n x_0),\ d(u, f^{n+1} x_0),\ d(f^n x_0, fu).$$

Therefore, as $n \to \infty$ the above inequality yields

$$(4) \qquad d(fu, u) \leqslant \varphi(d(u, fu), 0, 0, 0, d(u, fu)).$$

If $d(u, fu) = t > 0$ then by (4)

$$t \leqslant \varphi(t, t, t, t, t) < t,$$

a contradiction. Thus $fu = u$.

To prove uniqueness, suppose there is a $v \neq u$ for which $fu = u$ and $fv = v$. Let $r = d(u, v) > 0$. Then by (2)

$$r = d(u, v) = d(fu, fv) \leqslant \varphi(0, 0, r, r, r) < r,$$

contradicting $r > 0$. Thus $v = u$.

Corollary 1. *Suppose $f: X \to X$ satisfies either* (1) *or the condition: there exists nonnegative constants a, b, c with $2a + 2b + c < 1$ such that for all $x, y \in X$,*

$$(5) \qquad d(fx, fy) \leqslant a(d(x, fx) + d(y, fy)) + b(d(x, fy) + d(y, fx)) + cd(x, y)$$

Then f has a unique fixed point in X.

Proof. Since (5) implies (1) with $k = 2a + 2b + c$, it suffices to prove the result satisfying condition (1). Now, it follows (see Ćirić [1]) that mappings (1) also satisfy (3) for each $x \in X$. Further, defining $\varphi: (R^+)^5 \to R^+$ as

$$\varphi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_i, t_2, t_3, t_4, t_5\},$$

it is easy to verify that φ satisfies the conditions of Theorem 1. Thus f has a unique fixed point in X.

It may be remarked that several fixed point theorems have been obtained (see Hardy & Rogers [2], Kannan [3], Reich [4], Sehgal [5]) under condition (5) when some of the constants in (5) are zeros. All these results are special cases of (1) and hence of Theorem 1. Now, we give a simple example of a mapping f that satisfies (2) but not (1) for any value of $k < 1$.

EXAMPLE. Let $X = [0, \infty)$ with $d(x, y) = |x - y|$. Define a mapping $f: X \to X$ by

$$fx = \frac{x}{1 + x}$$

and let $\varphi: (R^+)^5 \to R^+$ be defined as

$$\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{t_3}{1 + t_3}.$$

Then it is easy to verify that φ satisfies all the conditions of Theorem 1. Furthermore, for any $x, y \in X$,

$$d(fx, fy)=\frac{|x-y|}{1+x+y+xy}\leqslant\frac{|x-y|}{1+|x-y|}=\varphi(|x-fx|, |y-fy|, |x-y|, |x-fy|\,|y-fx|)$$

Thus (2) holds. Since f satisfies (3) for each $x \in X$, therefore, Theorem 1 applies and in fact $f0=0$ is the unique fixed point of f in X. However, f does not satisfy (1), for otherwise there is a $k<1$ such that for all $x \in X$

$$\frac{x}{1+x}=d(f0, fx)\leqslant k\max\left\{0, \frac{x^2}{1+x}, x, \frac{x}{1+x}, x\right\}. \tag{6}$$

Since for any $x \in R^+$, $\frac{x^2}{1+x}\leqslant x$, it follows by (6) that for each $x\geqslant 1$, $\frac{x}{1+x}\leqslant kx$ that is $\frac{1}{1+x}\leqslant k$ for each $x\geqslant 1$. This is clearly impossible. Thus, f does not satisfy (1) for any value of $k<1$. Therefore, Ćirić's result (with Condition (1), Corollary 1) is in fact a special case of Theorem 1.

REFERENCES

[1] Ćirić, L. B., *A generalization of Banach's contraction principle*, Proc. Amer. Math Soc. 45 (2) (1974) 267—273.

[2] Hardy, G. E. & Rogers, T. D., *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. 16 (2) (1973) 201—206.

[3] Kannan, R., *Some remarks on fixed points*, Bull. Calcutta Math. Soc. 60 (1968) 71—76.

[4] Reich, S., *Kannan's fixed point theorems*, Boll. Unione Mat. Italiana, 4 (1971) 1—11.

[5] Sehgal, V. M., *Some fixed and common fixed point theorems in metric spaces* Canad. Math. Bull., 16 (4) (1974) 257—259.

Department of Mathematics
University of Wyoming
Laramie, Wyoming 82071
USA