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A FIXED POINT THEOREM FOR A CLASS OF MAPPINGS IN PROBABILISTIC LOCALLY CONVEX SPACES

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The notion of probabilistic locally convex spaces was introduced in [2] by V. Istratescu and we shall prove in this paper a fixed point theorem for a class of mappings of probabilistic locally convex spaces into itself. As a corollary we shall obtain Theorem 1 in [1] which was proved by Florea Gandac.

First, we shall give some notations and definitions which we shall use in the sequel.

Let S be a linear space over the real or complex field K and for every i in the index set I consider a function $\mathcal{F}^i: S \to \Delta^+$ where Δ^+ is the family of distribution functions F such that $F(0)=0$ (a distribution function F is a nondecreasing and leftcontinuous mapping of reals into $[0,1]$ with properties $\inf F(x)=0$, $\sup F(x)=1$). We shall denote $\mathcal{F}^i(x)$ by F_x^i.

D e f i n i t i o n. *S is called a probabilistic locally convex space iff for each $i \in I$ the following conditions are satisfied:*

1. $F_0^i = H$ *where* $H(x)=\begin{cases}0 & x \leqslant 0\\ 1 & x>0\end{cases}$.

2. $F_{\lambda x}^i(\varepsilon)=F_x^i\left(\dfrac{\varepsilon}{|\lambda|}\right)$ *for every* $\lambda \in K$, $\lambda \neq 0$; *every* $x \in S$ *and every* $\varepsilon>0$.

3. $F_{x+y}^i(\varepsilon_1+\varepsilon_2) \geqslant t(F_x^i(\varepsilon_1), F_y^i(\varepsilon_2))$ *for every* $x, y \in S$; $\varepsilon_1, \varepsilon_2>0$ *where the mapping* t: $[0,1]\times[0,1]\to[0,1]$ *satisfies the following conditions*:

I. $t(0,0)=0$, $t(a, 1)=a$ *for every* $a \in [0,1]$.

II. *If* $c \geqslant a$ *and* $d \geqslant b$ *then* $t(c, d) \geqslant t(a, b)$.

III. $t(a, b)=t(b, a)$ *for every* $a, b \in [0,1]$.

IV. $t(a, t(b, c))=t(t(a, b), c)$ *for every* $a, b, c \in [0,1]$.

We shall suppose in this paper that $t(a, b)=\min\{a, b\}$ for every $a, b \in [0,1]$ and it is easy to see that S becomes a locally convex space if

$$\mathcal{N}=\{N^i(\varepsilon, \xi)\}_{i \in I,\ \varepsilon>0,\ \xi \in (0,1)}$$

is the neighbourhood system at 0 where $N^i(\varepsilon, \xi)=\{x \mid F_x^i(\varepsilon)>1-\xi\}$. Further if $\{F_x^i=H \text{ for every } i \in I\} \Leftrightarrow \{x=0\}$ then S is a Hausdorff locally convex space.

Note that a net $\{x_d\}_{d\in D}$ converges to 0 if and only if for every $i\in I$, every $\varepsilon\in(0, \infty)$ and every $\xi\in(0,1)$ there exists $d_0(i, \varepsilon, \xi)\in D$ such that:

$$F^i_{x_d}(\varepsilon)>1-\xi, \text{ for every } d\geqslant d_0.$$

Theorem. *Let $(S, \mathcal{F}, \min)$(*) be a sequentially complete Hausdorff probabilistic locally convex space and M be a closed subset of S. Further, suppose that T is a continuous mapping from M into M such that the following conditions are satisfied:*

1. *For every $i\in I$ there exist $q(i)>0$ and $f(i)\in I$ such that for every $\varepsilon>0$, $x\in M$ and $y\in M$ the following inequality holds;*

$$F^i_{T^{n(x)}x-T^{n(x)}y}(q(i)\varepsilon)\geqslant F^{f(i)}_{x-y}(\varepsilon)$$

where $n(x)$ is a natural number which depends on x.

2. *For every $i\in I$ there exist $n(i)\in N$ and $Q(i)\in(0,1)$* such that *$q(f^n(i))\leqslant Q(i)$ for every $n\geqslant n(i)$.*

3. *There exists $x_0\in M$ such that for every $\varepsilon>0$ and $i\in I$.*

$$\lim_{p\to\infty} F^{f^n(i)}_{T^k x_0-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)=1$$

uniformly in respect to $n=0,1,\ldots$ for every $k=1, 2, \ldots, n(x_0)$. Then there exists one and only one solution x^* of the equation $Tx=x$ which satisfies the condition:

$$\lim_{p\to\infty} F^{f^n(i)}_{x^*-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)=1 \tag{1}$$

for every $i\in I$, $\varepsilon>0$ uniformly in respect to $n=0, 1, 2, \ldots$.

Proof: First, we shall prove that:

$$\lim_{p\to\infty} F^{f^n(i)}_{T^m x_0-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)=1 \tag{2}$$

for every $i\in I$ and $\varepsilon>0$ uniformly in respect to $m\in N$ and $n\geqslant n(i)$.

Let us suppose that $sn(x_0)<m\leqslant(s+1)\,n(x_0)$ where $s\in N$. Then from the condition 1. of the Theorem and the property 3. in the Definition it follows for $n\geqslant n(i)$:

$$F^{f^n(i)}_{T^m x_0-x_0}(\varepsilon)\geqslant\min\Big\{F^{f^n(i)}_{T^{m-n(x_0)}T^{n(x_0)}x_0-T^{n(x_0)}x_0}(q(f^n(i))\varepsilon),$$

$$,F^{f^n(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^n(i))))\Big\}\geqslant\min\Big\{F^{f^{n+1}(i)}_{T^{m-n(x_0)}x_0-x_0}(\varepsilon),$$

$$,F^{f^n(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^n(i))))\Big\}\geqslant\min\Big\{F^{f^{n+1}(i)}_{T^{m-2n(x_0)}T^{n(x_0)}x_0-T^{n(x_0)}x_0}(q(f^{n+1}(i))), (\varepsilon)$$

(*) If M is a probabilistic bounded subset of S and $F^{f^n(i)}_x(\varepsilon)\geqslant F^{g(i)}_x(\varepsilon)$, $g(i)\in I$ for every $i\in I$, $n\geqslant 0$, $x\in S$ and $\varepsilon>0$ it is easy to see that t can be any continuous T-norm (M is probabilistic bounded if, for every $i\in I$, $\sup_{\varepsilon}\sup_{\delta<\varepsilon}\inf_{x,y\in M} F^i_{x-y}(\delta)=1$)

$$,F^{f^{n+1}(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^{n+1}(i)))),$$

$$\left.,F^{f^{n}(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^{n}(i))))\right\} \geqslant \min\left\{F^{f^{n+2}(i)}_{T^{m-2n(x_0)}x_0-x_0}(\varepsilon),\right.$$

$$,F^{f^{n+1}(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^{n+1}(i)))),$$

$$\left.,F^{f^{n}}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^{n}(i))))\right\} \geqslant \cdots \geqslant \min\left\{F^{f^{n+s}(i)}_{T^{m-n(x_0)s}x_0-x_0}(\varepsilon),\right.$$

$$\left.,F^{f^{n+s-1}}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^{n+s-1}(i)))), \ldots, F^{f^{n}(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^{n}(i))))\right\}$$

Because of $q(f^n(i)) \leqslant Q(i) < 1$ for every $n \geqslant n(i)$ we have:

$$F^{f^{n}(i)}_{T^{m}x_0-x_0}(\varepsilon) \geqslant \min\left\{F^{f^{n+s}(i)}_{T^{m-sn(x_0)}x_0-x_0}(\varepsilon),\right.$$

$$\left.F^{f^{n+s-1}(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^{n+s-1}(i)))), \ldots, F^{f^{n}(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-q(f^{n}(i))))\right\} \geqslant$$

$$\geqslant \min\left\{F^{f^{n+s}(i)}_{T^{m-sn(x_0)}x_0-x_0}(\varepsilon),\right.$$

$$\left.,F^{f^{n+s-1}(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-Q(i))), \ldots, F^{f^{n}(i)}_{T^{n(x_0)}x_0-x_0}(\varepsilon(1-Q(i)))\right\}$$

Further from the condition 3. of the Theorem it follows that for every $i \in I$, $\varepsilon > 0$ and $\xi \in (0, 1)$ there exists $P(i, \varepsilon, \xi)$ such that:

$$F^{f^{n}(i)}_{T^{k}x_0-x_0}\left(\frac{\varepsilon'}{Q(i)^p}\right) > 1-\xi, \quad \varepsilon' = \varepsilon(1-Q(i))$$

for every $p \geqslant P(i, \varepsilon, \xi)$, $n = 0, 1, 2, \ldots;$ $k = 1, 2, \ldots, n(x_0)$.

So we have:

$$F^{f^{n}(i)}_{T^{m}x_0-x_0}\left(\frac{\varepsilon'}{Q(i)^p}\right) > \min\{1-\xi, 1-\xi, \ldots, 1-\xi\} = 1-\xi$$

for every $p \geqslant P(i, \varepsilon, \xi)$, $m > n(x_0)$, $n \geqslant n(i)$ which means;

$$\lim_{p \to \infty} F^{f^{n}(i)}_{T^{m}x_0-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1$$

uniformly in respect to $n \geqslant n(i)$ and $m \in N$.

Let us denote $n(x_0)$ by m_0 and define the sequence $\{x_n\}$ in the following way:

$$x_1 = T^{m_0}x_0; \; m_i = n(x_i); \; x_{i+1} = T^{m_i}x_i \quad i = 1, 2, \ldots$$

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We shall prove that the sequence $\{x_n\}$ is a Cauchy sequence i.e. that for every $i \in I$, $\varepsilon > 0$ and $\xi \in (0, 1)$ there exists $N(i, \varepsilon, \xi)$ such that:

$$F^i_{x_{n+p}-x_n}(\varepsilon) > 1 - \xi, \quad n \geq N(i, \varepsilon, \xi) \text{ and } p = 1, 2, \dots .$$

From the condition 1. of the Theorem it follows for $n \geq n(i) + 1$:

$$F^i_{x_{n+p}-x_n}(\varepsilon) = F^i_{T^{m_{n+p-1}}T^{m_{n+p-2}}\dots T^{m_{n-1}}x_{n-1} - T^{m_{n-1}}x_{n-1}}(\varepsilon) \geq$$

$$\geq F^{f(i)}_{T^{m_{n+p-1}}\dots T^{m_n}x_{n-1} - x_{n-1}}\left(\frac{\varepsilon}{q(i)}\right) \geq \dots \geq$$

$$\geq F^{f^n(i)}_{T^{\sum_{s=n}^{n+p-1} m_s} x_0 - x_0}\left(\frac{\varepsilon(i)}{Q(i)^n}\right) \quad \text{where } \varepsilon(i) = \frac{Q(i)^{n(i)}\varepsilon}{\prod_{r=0}^{n(i)-1} q(f^r(i))}$$

From the relation (2) we conclude that for every $i \in I$, $\varepsilon > 0$ and $\xi \in (0, 1)$ there exists $N(i, \varepsilon, \xi)$ such that:

$$F^{f^n(i)}_{T^{\sum_{s=n}^{n+p-1} m_s} x_0 - x_0}\left(\frac{\varepsilon(i)}{Q(i)^n}\right) > 1 - \xi \text{ for every } n \geq N(i, \varepsilon, \xi)$$

and $p = 1, 2, \dots$. and so the sequence $\{x_n\}$ is a Cauchy sequence. Because the space S is sequentially complete it follows that there exists $\lim_{n\to\infty} x_n = x^*$ and we shall prove that $Tx^* = x^*$.

We have;

$$F^i_{Tx_n - x_n}(\varepsilon) = F^i_{TT^{m_{n-1}}x_{n-1} - T^{m_{n-1}}x_{n-1}}(\varepsilon) \geq F^{f(i)}_{Tx_{n-1}-x_{n-1}}\left(\frac{\varepsilon}{q(i)}\right) \geq \dots \geq$$

$$\geq F^{f^n(i)}_{Tx_0 - x_0}\left(\frac{\varepsilon}{\prod_{s=0}^{n-1} q(f^s(i))}\right) \quad \text{and so } \lim_{n\to\infty} Tx_n - x_n = 0.$$

This implies: $\lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n$, because the mapping T is continuous, and so we have $Tx^* = x^*$.

Now we shall show that for every $i \in I$ and $\varepsilon > 0$:

$$\lim_{p\to\infty} F^{f^n(i)}_{x^* - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1$$

uniformly in respect to $n = 0, 1, 2, \dots$. First, suppose that $n \geq n(i)$. Then for every $\varepsilon > 0$, $i \in I$ and $\xi \in (0, 1)$ there exists $P(i, \varepsilon, \xi)$ such that:

$$F^{f^n(i)}_{T^m x_0 - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) > 1 - \xi \text{ for every } p \geq P(i, \varepsilon, \xi),$$

every $n \geqslant n(i)$ and every $m \in N$. Because of $x_n = T^{\sum_{s=0}^{n-1} m_s} x_0$ it follows:

$$F^{f^r(i)}_{x_n - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) > 1 - \xi \text{ for every, } p \geqslant P(i, \varepsilon, \xi)$$

every $n \in N$ and $r \geqslant n(i)$ and from inequality:

$$F^{f^r(i)}_{x^* - x_n}\left(\frac{\varepsilon}{Q(i)^p}\right) \geqslant \min\left\{F^{f^r(i)}_{x^* - x_n}\left(\frac{\varepsilon}{2\,Q(i)^p}\right),\ F^{f^r(i)}_{x_n - x_0}\left(\frac{\varepsilon}{2\,Q(i)^p}\right)\right\}$$

we obtain:

$$F^{f^r(i)}_{x^* - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) > 1 - \xi,\ p \geqslant P'(i, \varepsilon, \xi) \text{ for every } r \geqslant n(i).$$

On the other hand for every $i \in I$, $\lim_{p \to \infty} F^{f^r(i)}_{x^* - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1$ for $r = 0, 1, \ldots,$ $n(i) - 1$ and so there exists $P''(i, \varepsilon, \xi)$ such that: $F^{f^r(i)}_{x^* - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) > 1 - \xi$ for $p \geqslant P''$, $r \in N$ i.e. $\lim_{p \to \infty} F^{f^r(i)}_{x^* - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1$ uniformly in respect to $r = 0, 1, 2, \ldots.$ Now, suppose that $y = Ty$ and for every $i \in I$, $\varepsilon > 0$:

$$\lim_{p \to \infty} F^{f^n(i)}_{y - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1$$

uniformly in respect to $n = 0, 1, 2, \ldots.$ and prove that $x^* = y$. We have:

$$F^i_{x^* - y}(\varepsilon) = F^i_{T^{n(x^*)}x^* - T^{n(x^*)}y}(\varepsilon) \geqslant F^{f(i)}_{x^* - y}\left(\frac{\varepsilon}{q(i)}\right) \geqslant \cdots \geqslant$$

$$\geqslant F^{f^n(i)}_{x^* - y}\left(\frac{\varepsilon}{\prod_{r=0}^{n-1} q(f^r(i))}\right) \geqslant \min\left\{F^{f^n(i)}_{x^* - x_0}\left(\frac{\varepsilon}{2 \cdot \prod_{r=0}^{n-1} q(f^r(i))}\right),\right.$$

$$\left. F^{f^n(i)}_{y - x_0}\left(\frac{\varepsilon}{2 \cdot \prod_{r=0}^{n-1} q(f^r(i))}\right)\right\} \quad \text{and if } n \to \infty \text{ we obtain:}$$

$$F^i_{x^* - y}(\varepsilon) = 1, \text{ for every } \varepsilon > 0 \text{ and } i \in I \text{ so } x^* = y.$$

Let us show that $x^* = \lim_{n \to \infty} T^n x_0$. Suppose that $n \geqslant n(x^*)$ and $n = rn(x^*) + p$, $0 \leqslant p < n(x^*)$. Then we have for $r \geqslant n(i) + 1$:

$$F^i_{T^n x_0 - x^*}(\varepsilon) = F^i_{T^{rn(x^*)+p} x_0 - T^{n(x^*)} x^*}(\varepsilon) \geqslant F^{f(i)}_{T^{(r-1)\,n(x^*)+p} x_0 - x^*}\left(\frac{\varepsilon}{q(i)}\right) \geqslant$$

$$\geqslant F^{f^2(i)}_{T^{(r-2)\,n(x^*)+p} x_0 - x^*}\left(\frac{\varepsilon}{q(i)\,q(f(i))}\right) \geqslant \cdots \geqslant$$

$$\geqslant F^{f^r(i)}_{T^p x_0 - x^*}\left(\frac{\varepsilon}{\prod_{s=0}^{r-1} q(f^s(i))}\right) \geqslant \min\left\{F^{f^r(i)}_{T^p x_0 - x_0}\left(\frac{\varepsilon}{2\cdot\prod_{s=0}^{n(i)-1} q(f^s(i))\,Q(i)^{r-n(i)}}\right),\right.$$

$$\left.F^{f^r(i)}_{x^*-x_0}\left(\frac{\varepsilon}{2\cdot\prod_{s=0}^{n(i)-1} q(f^s(i))\,Q(i)^{r-n(i)}}\right)\right\} = \min\left\{F^{f^r(i)}_{T^p x_0 - x_0}\left(\frac{\varepsilon'(i)}{Q(i)^r}\right),\right.$$

$\left.F^{f^r(i)}_{x^*-x_0}\left(\frac{\varepsilon(i)}{Q(i)^r}\right)\right\}$. Let $R(i, \varepsilon, \xi)$ be such natural number that $r \geqslant R(i, \varepsilon, \xi)$ implies:

$$F^{f^r(i)}_{T^p x_0 - x_0}\left(\frac{\varepsilon(i)}{Q(i)^r}\right) > 1 - \xi \text{ and}$$

$$F^{f^r(i)}_{x^*-x_0}\left(\frac{\varepsilon(i)}{Q(i)^r}\right) > 1 - \xi.$$

Then we have $F^i_{T^n x_0 - x^*}(\varepsilon) > 1 - \xi$ if $n \geqslant N(i, \varepsilon, \xi)$ where $N(i, \varepsilon, \xi) = n(x^*) + n(x^*) R(i, \varepsilon, \xi)$.

Corollary. *Let T be a continuous mapping of a closed subset M of a Hausdorff sequentially complete locally convex space E into itself. Suppose that the following conditions are satisfied:*

1. *For every $i \in I$ there exist $q(i) > 0$ and $f(i) \in I$ such that the following inequality holds:*

$$|T^{n(x)} x - T^{n(x)} y|_i \leqslant q(i) \, |x - y|_{f(i)}$$

for every $x, y \in M$ where $\{|\ |_i\}_{i \in I}$ is the family of seminorms defining the topology in E and $n(x)$ is a natural number which depends on x.

2. *For every $i \in I$ $\overline{\lim_{n\to\infty}}\, q(f^n(i)) < 1$.*

3. *There exists $x_0 \in M$ such that:*

$$\sup |x_0 - T^k x_0|_{f^n(i)} < \infty \quad k = 1, 2, \ldots, n(x_0) \quad n = 0, 1, 2, \ldots$$

Then there exists only one solution of the equation $x = Tx$ which also satisfies the condition:

$$\sup |x^* - x_0|_{f^n(i)} < \infty \quad n = 0, 1, 2, \ldots$$

Proof: Let $F_x^i(\varepsilon) = \begin{cases} 1 & |x|_i < \varepsilon \\ 0 & |x|_i \geqslant \varepsilon \end{cases}$ for every $x \in E$, $i \in I$ and $t(a, b) = \min\{a, b\}$. It is easy to see that $(E, \mathcal{F}, \min)$ is a sequentially complete Hausdorff locally convex space. First, we shall show that:

$$F^i_{T^{n(x)}x - T^{n(x)}y}(q(i)\,\varepsilon) \geqslant F^{f(i)}_{x-y}(\varepsilon) \text{ for every } \varepsilon > 0,$$

every $i \in I$ and every $x, y \in M$. Suppose now that $F^{f(x)}_{x-y}(\varepsilon) = 1$ i.e. that $|x - y|_{f(i)} < \varepsilon$. Then we have:

$$|T^{n(x)} x - T^{n(x)} y|_i < q(i)\,\varepsilon$$

which means that $F^{i}_{T^{n}(x)x-T^{n}(x)y}(q(i)\varepsilon)=1=F^{f(i)}_{x-y}(\varepsilon)$. Now, we shall prove that:

$$\lim_{p\to\infty} F^{f^{n}(i)}_{T^{k}x_0-x_0}\left(\frac{\varepsilon}{Q(i)^{p}}\right)=1 \qquad k=1,2,\ldots,n(x_0)$$

uniformly in respect to $n=0, 1, 2, \ldots$. i.e. that for every $i\in I$, $\varepsilon>0$ and $\xi\in(0, 1)$ there exists $P(i, \varepsilon, \xi)$ such that:

$$1=F^{f^{n}(i)}_{T^{k}x_0-x_0}\left(\frac{\varepsilon}{Q(i)^{p}}\right) \text{ for every } p\geqslant P(i, \varepsilon, \xi),$$

every $n=0, 1, 2, \ldots$ and every $k=1, 2, \ldots, n(x_0)$. So, we must prove that there exists $P(i, \varepsilon, \xi)$ such that:

$$|T^{k}x_0-x_0|_{f^{n}(i)}\leqslant\frac{\varepsilon}{Q(i)^{p}}, \text{ for every } p\geqslant P(i, \varepsilon, \xi)$$

$n=0, 1, 2, \ldots$ and $k=1, 2, \ldots, n(x_0)$. Let us denote $\sup|T^{k}x_0-x_0|_{f^{n}(i)}$ $k=1, 2, \ldots, n(x_0)$ $n=0, 1, 2, \ldots$ by $M(i)$.

Because of $Q(i)\in(0, 1)$ it is easy to see that there exists $P(i, \varepsilon, \xi)$ such that:

$$M(i)\,Q(i)^{p}\leqslant\varepsilon \text{ for every } p\geqslant P(i, \varepsilon, \xi) \text{ q.e.d.}$$

REFERENCES

[1] Florea Gandac, *Teoreme de punct fix in spatii locall convexe*, Stud. si cerc. mat. Tom 24, Nr. 7, 1972, pp. 1097—1106.

[2] V. I. I. Istratescu, *Introducere in teorta spatiilor metrice probabiliste cu aplicatii*, Bucuresti, 1974.

[3] V. Radu, *Locally convex probabilistic spaces and linear operators*, Universitatea din Timisoara, Seminarul de teorija functiilor si matematici aplicate, A. Spatii matrice probabiliste, 22, 1974.

[4] O. Hadžić, B. Stanković, *Some theorems on the fixed point in locally convex space*, Publ. Inst. Math. Beograd, 10 (24), 9—19, 1970.

[5] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. 10, 1960, 313—334.