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ON TWO CLASSES OF WEIGHTED SHIFTS

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Introduction.

An operator T in Hilbert space H is called a (unilateral) weighted shift (abbreviated: w. s.) iff

$$Te_n = \alpha_n e_{n+1} \qquad (n=0, 1, 2, \ldots),$$

where $\{e_n\}_0^\infty$ is an orthonormal basis in H and $\{\alpha_n\}_0^\infty$ a bounded sequence of positive numbers. In the case $\alpha_n = 1$, $n=0, 1, 2, \ldots$, T is called a shift and denoted by S.

We will say that T belongs to the class E (to the class F) iff every nontrivial part of T (i. e. restriction to a nonzero invariant subspace for T) is unitarily equivalent (similar) to a w. s.

One can pose the problem of the description of these classes. According to the classical Beurling's result [1], $S \in E$. In Section I we will show (Theorem 1.) that, if

$$(1) \qquad l = \liminf_{n\to\infty} (\alpha_0 \alpha_1 \cdots \alpha_{n-1})^{\frac{1}{n}} = \liminf_{n\to\infty} (w_n)^{\frac{1}{n}} > 0 \qquad (w_0 = 1),$$

then the class E consists of w. shifts for which all α's, except α_0, are equal. Halmos has asked in [2], Problem 2, whether every w. s. T belongs to the class F. Gellar, [3], has showed that there exists w. s. outside of F. Theorem 2. in Section II of this paper provides a further information about the class (again under assumption of (1)): if $T \in F$, then a part of T is similar to T itself.

Section I. Lemma 1. *If an isometry A realizes the unitary equivalence of a part T/M, of a w.s. T to some w. s. R, $Re_n = \beta_n e_{n+1}$, i. e. $AM = H$ and $T/M = A^{-1}RA$, and if $Pe_0 \neq 0$ (where P is the projector from H to M), then the vector $A^{-1}e_0$ is a scalar multiple of Pe_0.*

Proof. Since A is an isometry, the sequence

$$(2) \qquad \{T^n A^{-1} e_0 = A^{-1} R^n e_0\}_0^\infty$$

is orthogonal and complete in M. The assertion of the Lemma 1. now follows from

$$e_0 \perp T^n A^{-1} e_0, \quad n = 1, 2, \ldots.$$

Theorem 1. *Let T be a w.s. and let* (1) *hold. Then,* $T \in E$ *iff*

(3) $$\alpha_i = \alpha_j \text{ for } i, j = 1, 2, \ldots .$$

Proof. Let $T \in E$. For $|\lambda| < l$, the vector

$$v = \sum_0^\infty \frac{\lambda^n}{w_n} e_n$$

is an eigenvector for T^*. The subspace $M = v^\perp$ is invariant for T. If $\lambda \neq 0$, then $Pe_0 \neq 0$. From $e_0 - Pe_0 \perp M$, it follows that $e_0 - Pe_0 = \gamma v$, where,

$$\gamma \cdot \sum_0^\infty \frac{|\lambda|^{2n}}{w_n^2} = 1.$$

By Lemma 1. and by orthogonality of the sequence (2), the sequence

$$\{T^n Pe_0\}_0^\infty$$

is orthogonal, too. By a routine computation, we obtain therefore

$$0 = \langle TPe_0, T^2 Pe_0 \rangle = \gamma^2 \lambda \sum_1^\infty \frac{\alpha_1{}^2 - \alpha_{n+1}^2}{w_n^2} |\lambda|^{2n},$$

giving

(4) $$\sum_1^\infty \frac{\alpha_1{}^2 - \alpha_{n+1}^2}{w_n^2} |\lambda|^{2n-2} \equiv 0.$$

for all λ, $0 < |\lambda| < l$. Hence, if $|\lambda| \to 0$, it follows that $\alpha_1{}^2 - {}_2{}^2 = 0$. Similarly, by putting $\alpha_1{}^2 - {}_2{}^2 = 0$ in (4), we can obtain $\alpha_1{}^2 - \alpha_3{}^3 = 0$, and so on, thus (3) holds.

Assume now that (3) holds. This. condition, evidently, can be replaced by

(3') $$\alpha_n = 1, \; n = 1, 2, \ldots ,$$

without any loss of generality. Let $H_0 = e_0^\perp$. Because of (3'), $T/_{H_0}$ is a shift. By [1], if M_0 is an invariant subspace for T/H_0, then there exists an isometry A from M_0 to H_0, for which

(5) $$T/M_0 = A^{-1}(T/H_0) A.$$

Let M be an arbitrary invariant subspace for T. If $M \perp e_0$, then $M \subset H_0$, and the assertion holds. If M is not orthogonal to e_0, then $M_0 = M \cap H_0$ is an invariant subspace for T/H_0; this means that there exists an isometry A for which (5) is satisfied. Hence, the sequence

(6) $$\left\{ A^{-1} e_n = \frac{1}{\alpha_0} A^{-1} T^{n-1} e_1 = \frac{1}{\alpha_0} T^{n-1} A^{-1} e_1 \right\}_1^\infty$$

is an orthonormal basis in M_0. Let Pe_0 be the projection of e_0 to M. We will show that, after addition of Pe_0 to the sequence (6), the sequence becomes an orthogonal basis (6') in M. It will suffice to show that $x \in M$ and $x \perp M_0$ imply Tx to be a scalar multiple of $A^{-1} e_1$. This means, because of $\ker T = 0$, that the subspace $M \ominus M_0$ is one-dimensional. But, if $x \in M$ and $x \perp M_0$, then

$Tx = Sx + \langle x, e_0\rangle (\alpha_0 - 1) e_1 \perp SM_0 = TM_0$ and $Tx \in M_0$, which, together with the completeness of the sequence (6), implies the colinearity of Tx and $A^{-1} e_1$. Thus, the sequence (6') is complete in M.

In the system

(7) $$\{T^n Pe_0\}_0^\infty$$

each vector, except Pe_0, is a scalar multiple of the corresponding vector in (6). Therefore the system (7) is orthogonal and complete in M. To complete the proof, it is only necessary to normalize the sequence (7) and to obtain in such a way an o.n. basis on which $^T/_M$ acts as a w.s.

Without (1), Theorem 1. does not always hold. There exist examples of w.s. in the class E, for which neither (1) nor (2) is satisfied [4].

Section II. We will use the following two lemmas for proving the Theorem 2. Lemma 3. appears to be of interest by itself.

Lemma 2. *Let a (non-trivial) part,* $^T/_M$, *of a w.s.* T *be similar to some w.s.* R, $Re_n = \beta_n e_{n+1}$, *and let* P_n *denote the projector to the subspace* $T^n M$ $(n = 0, 1, 2, \ldots)$. *(By* $T^n M$ *we denote the closed subspace in* M *generated by* $T^n M$*). If* $P_0 e_0 \neq 0$, *then*

a) *the sequence*

(8) $$\{P_n e_n\}_0^\infty \text{ is orthogonal;}$$

b) *the sequence* $\{P_i e_i\}_0^n$ *is complete in* $\ker (^A/_M)^{*\,n+1}$; *for* $n = 0, 1, 2, \ldots$;

c) *the sequence* (8) *is complete in* M.

Proof. a) Orthogonality follows from $P_n e_n \perp T^m M$ (because of $e_n \perp T^m M$) for $m > n$ and from $P_m e_m \in T^m M$.

b) Since the sequence $\{e_i\}_0^n$ is complete in $\ker R^{*n}$ and since the dimension of the kernel of an operator is a similarity invariant, it follows that $\dim (\ker (^T/_M)^{*\,n+1}) = n + 1$, and the statement holds.

c) Let $x \in M$ and $x \perp P_n e_n$, $n = 0, 1, 2, \ldots$. It follows from $x \perp Pe_0$ that $x \perp e_0$ and $x \in AM$ (by b). Then $x \in AM$ and $x \perp P_1 e_1$ imply $x \perp e_1$ (and $x \in A^2 M$), and so forth, i.e. $x \perp e_n$ for $n = 0, 1, 2, \ldots$; thus $x = 0$.

Lemma 3. *Take for w.s.* T *and* R *and for the subspace* M *the same assumptions as in Lemma 2. Denote by* k *the smallest index for which* $P_0 e_k \neq 0$. *Then the sequence*

(9) $$\left\{Q_n = \frac{\alpha_k \alpha_{k+1} \cdots \alpha_{k+n}}{\beta_0 \beta_1 \cdots \beta_n \| P_{n+1} e_{k+n+1} \|}\right\}_0^\infty$$

is bounded away from 0 *and from* ∞.

Proof. Explicit proof of the lemma is restricted to the case $k = 0$. (If $k > 0$, T can be replaced by $T' : T'e_n = \alpha_{k+n} e_{n+1}$, which is unitarily equivalent to T).

Let an invertible operator A (acting from H to M) realize the similarity of R and $^T/_M$, i.e. $R=A^{-1}(^T/_M)A$. Put

$$f_n=\frac{P_n e_n}{\|P_n e_n\|} \qquad (n=0,1,2,\ldots).$$

Since

$$A^{*-1}R^*=(^T/_M)^* A^{*-1},$$

we have

$$\beta_n\langle A^{*-1}e_n, f_n\rangle=\langle A^{*-1}R^*e_{n+1}, f_n\rangle= \tag{10}$$

$$=\langle(^T/_M)^* A^{*-1}e_{n+1}, f_n\rangle=\langle A^{*-1}e_{n+1}, Tf_n\rangle.$$

Relation $e_{n+1}\in\ker R^{*n+2}$ implies $A^{*-1}e_{n+1}\in\ker(^T/_M)^{*n+2}$. By Lemma 2. b), $A^{*-1}e_{n+1}$ is a linear combination of $\{f_i\}_0^{n+1}$. But $Tf_n\in T^{n+1}M$ and $Tf_n\perp f_i$, $i=0,1,2,\ldots,n$, so that the last inner product in (10) is equal to

$$\langle A^{*-1}e_{n+1}, f_{n+1}\rangle\langle f_{n+1}, Tf_n\rangle.$$

Since

$$\langle f_{n+1}, Tf_n\rangle=\frac{1}{\|P_{n+1}e_{n+1}\|}\langle e_{n+1}, Tf_n\rangle=\frac{1}{\|P_{n+1}e_{n+1}\|}\langle T^*e_{n+1}, f_n\rangle=$$

$$=\frac{\alpha_n}{\|P_{n+1}e_{n+1}\|}\langle e_n, f_n\rangle=\frac{\alpha_n}{\|P_{n+1}e_{n+1}\|}\langle P_n e_n, f_n\rangle=\alpha_n\frac{\|P_n e_n\|}{\|P_{n+1}e_{n+1}\|},$$

the relation (10) implies

$$\beta_n\langle A^{*-1}e_n, f_n\rangle=\alpha_n\frac{\|P_n e_n\|}{\|P_{n+1}e_{n+1}\|}\langle A^{*-1}e_{n+1}, f_{n+1}\rangle,$$

i.e

$$\langle A^{*-1}e_{n+1}, f_{n+1}\rangle=\frac{\beta_n\beta_{n-1}\cdots\beta_0}{\alpha_n\alpha_{n-1}\cdots\alpha_0}\,\frac{\|P_{n+1}e_{n+1}\|}{\|P_0 e_0\|}\langle A^{*-1}e_0, f_0\rangle.$$

Hence we conclude directly that the sequence (9) is bounded away from 0:

$$Q_n^{-1}=\left|\langle A^{*-1}e_{n+1}, f_{n+1}\rangle\frac{\|P_0 e_0\|}{\langle A^{*-1}e_0, f_0\rangle}\right|\leqslant$$

$$\leqslant\|A^{*-1}\|\frac{\|P_0 e_0\|}{|\langle A^{*-1}e_0, f_0\rangle|} \qquad (n=0,1,2,\ldots).$$

In the similar way, starting with $AR=(^T/_M)A$, we obtain

$$\beta_n\langle Ae_{n+1}, f_{n+1}\rangle=\langle ARe_n, f_{n+1}\rangle=\langle TAe_n, f_{n+1}\rangle=$$

$$=\langle TAe_n, e_{n+1}\rangle\frac{1}{\|P_{n+1}e_{n+1}\|}=\langle Ae_n, T^*e_{n+1}\rangle\frac{1}{\|P_{n+1}e_{n+1}\|}=$$

$$=\langle Ae_n, e_n\rangle\frac{\alpha_n}{\|P_{n+1}e_{n+1}\|}=\alpha_n\frac{\|P_n e_n\|}{\|P_{n+1}e_{n+1}\|}\langle Ae_n, f_n\rangle$$

and

$$\langle Ae_{n+1}, f_{n+1}\rangle = \frac{\alpha_n \alpha_{n-1} \cdots \alpha_0}{\beta_n \beta_{n-1} \cdots \beta_0} \frac{\| P_0 e_0 \|}{\| P_{n+1} e_{n+1} \|} \langle Ae_0, f_0\rangle.$$

Thus, the sequence (9) is bounded:

$$Q_n = |\langle Ae_{n+1}, f_{n+1}\rangle| \frac{1}{\| P_0 e_0 \| \, |\langle Ae_0, f_0\rangle|} \leq \| A \| \frac{1}{\| P_0 e_0 \| \, |\langle Ae_0, f_0\rangle|}.$$

Theorem 2. *Suppose that for a w.s. T in F (1) holds. Then there exists a part of T similar to T itself.*

Proof. Let $0<|\lambda|<l$,

$$v = \sum_0^\infty \frac{\lambda^n}{w_n} e_n \qquad (v \neq e_0)$$

and $M = v^\perp$. We prove that for such an invariant subspace M the sequence $\{\| P_n e_n \|\}_0^\infty$ is bounded away from 0.

As in the proof of Theorem 1., we have $e_0 - P_0 e_0 = \gamma \cdot v$, where $\gamma = \frac{1}{\| v \|^2}$.

Thus $\| P_0 e_0 \|^2 = 1 - \frac{1}{\| v \|^2}$. Since

$$\| v \|^2 = \sum_0^\infty \frac{|\lambda|^{2n}}{w_n^2} > 1 + \frac{|\lambda|^2}{\alpha_0^2} \geq 1 + \frac{|\lambda|^2}{\| T \|^2},$$

it follows

$$\| P_0 e_0 \|^2 > \frac{|\lambda|^2}{\| T \|^2 + |\lambda|^2}.$$

Put now

$$v_1 = e_1 + \sum_1^\infty \frac{\lambda^n}{\alpha_1 \cdots \alpha_n} e_{n+1}.$$

It is quite easy to see that $x \in TM$ is equivalent to $x \perp v_1 \wedge x \perp e_0$. Namely, it follows from $x \perp v_1$ and $x \perp e_0$, because of $v = \frac{1}{\alpha_0} v_1 + e_0$, that $x \in M$, and $x \perp P_0 e_0$, too. Since $T/_M$ is similar to some w.s. R, the subspace $M \ominus TM$ is one-dimensional, thus $x \in TM$. In the same way as in the first step, we can obtain now

$$\| P_1 e_1 \| > \frac{|\lambda|^2}{\alpha_1^2 + |\lambda|^2} \geq \frac{|\lambda|^2}{\| T \|^2 + |\lambda|^2},$$

and

$$(11) \qquad (1>) \| P_n e_n \|^2 > \frac{|\lambda|^2}{\| T \|^2 + |\lambda|^2}, \qquad n = 0, 1, 2, \ldots.$$

If $\{\beta_n\}_0^\infty$ are weights of the w.s. R, to which $T/_M$ is similar, then, by Lemma 3. and by (11), the sequence

$$\left\{ \frac{\alpha_0 \alpha_1 \cdots \alpha_n}{\beta_0 \beta_1 \cdots \beta_n} \right\}_0^\infty$$

is bounded away from 0 and ∞, which implies that R is similar to T ([5], Problem 76.). That is, $^{T}/_{M}$ is similar to T.

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