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SOME QUESTIONS CONCERNING FOUNDATIONS OF MECHANICS

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1. Introduction

This paper is a continuation of the paper [4]. In it we shall be concerned with certain further questions concerning the foundations of mechanics. The questions which we shall concern here are the motion and some related concepts. We shall define them and specify laws of motion. The laws in question correspond to those of Newton. It means, as well, that mechanics which we here deal with corresponds to that of Newton.

Although we consider here only that part of mechanics which corresponds to Newtonian mechanics, we must say that this approach allows some further generalizations and the possibility of inclusion of a wider part of mechanics, as for instance continiuum mechanics. We shall show this in a forthcoming paper.

2. Fundamental concepts of mechanics

In this section we shall define the concept of a fundamental mechanical world and study the action of external forces upon it. We shall restrict our considerations to a fixed level of the mechanical universe $\boldsymbol{U}$ for which we shall suppose to consist of objects and forces of various levels. We shall be concerned with the universe $\boldsymbol{U}$ in more details in a forthcoming paper. Here we shall only assume that the level under consideration is that one consisting of objects we are surrounded by and forces between these objects.

As well as in the paper [4], we shall start from the mechanical system $\boldsymbol{M}$ consisting of a collection of mechanical objects, a collection of forces between these objects and possessing the structure of a fundamental semigroupoid [3]. It means that $\boldsymbol{M}$ is a system $\langle \boldsymbol{M}; \mathcal{D}_0, \mathcal{D}_1, \mathcal{C}\rangle$, where $\boldsymbol{M}$ means the collection of objects and forces, $\mathcal{D}_0$ and $\mathcal{D}_1$ mean two unary operations and $\mathcal{C}$ a binary operation on $\boldsymbol{M}$. The specification and meaning of these concepts are given in the paper [4].

Such a mechanical system is our start. We shall denote by $\boldsymbol{M}_{ob}$ the collection of objects and by $\boldsymbol{M}_{for}$ the collection of forces of it. Otherwise, we denote by For $(\ ,a)$ the set of all forces of $\boldsymbol{M}_{for}$ acting on the object $a \in \boldsymbol{M}_{ob}$

in a common point; and if all forces of this set have the same source, the object $b \in M_{ob}$ for instance, then we denote it by For(b, a). Thus For(b, a) is the set of all forces between objects b and a in $\boldsymbol{M}$.

In the paper [4], we have given necessary and sufficient conditions for an object of M upon which a set of forces of the form $\mathcal{F} = \{e, f_1, f_2, \ldots, f_n\}$ acts to be in equilibrium. We have: *an object of* $\boldsymbol{M}$ *will be in equilibrium under the action of a set of forces* $\mathcal{F}$ *if and only if this set has the group structure.* Considering the whole system $\boldsymbol{M}$, it is necessary and sufficient for the internal equilibrium that the system $\boldsymbol{M}$ has the structure of a groupoid [3].

Moreover we have seen that the problem of equilibrium of $\boldsymbol{M}$ under the action of forces, the sources of which are outside $\boldsymbol{M}$, we can consider in the same way extending the system $\boldsymbol{M}$ to a system $\boldsymbol{M}^*$ in such a way that the resultant of the forces acting upon an object a of $\boldsymbol{M}$ is contained in For($,a$) with respect to the new system $\boldsymbol{M}^*$. Certainly, we here assume that the source of that resultant is in $\boldsymbol{M}$.

Now we shall begin with the consideration of behaviour of $\boldsymbol{M}$ and its objects under the action of forces the sources of which are outside $\boldsymbol{M}$; forces with this property are external forces; we assume them to be unbalanced. Certainly, $\boldsymbol{M}$ and its objects will move under the action of such forces. In the sequel we shall be concerned with the motion of $\boldsymbol{M}$ and with laws which govern this motion.

However to be capable of this we have to involve certain further specifications into $\boldsymbol{M}$. First we assume that objects of $\boldsymbol{M}$ are materialized mathematical objects: it means mathematical objects provided with certain characteristics which will differentiate them from mathematical objects. Denote by $\mathcal{M}$ a set of material quantities which mechanical objects on the considered level of the mechanical universe can be supposed to possess. Assume that this set has the structure of a monoid. Then we define the concept of materialization as follows:

Definition 1. By the *materialization* of a mathematical object G we mean a morphism

$$\mu : \mathcal{M} \to \operatorname{Aut} G,$$

where Aut G means the group of automorphisms of G, such that

$$\mu(m)\, x = mx \in G,$$

and

$$(m_1 m_2)\, x = m_1 (m_2 x) \text{ and } 1\, x = x.$$

A pair (G, μ) consisting of a mathematical object G and the materialization μ we shall call a *mechanical object*.

If we have a mechanical object (G, μ), then we can define certain subobjects of it with respect to the action of $\mathcal{M}$. These subobjects are material particules. By a *material particule* of (G, μ) we mean a subobject consisting for a fixed $m \in \mathcal{M}$ of elements mx, $x \in G$. Hence, material particules of (G, μ) are its homogenous parts with respect to quantities of $\mathcal{M}$. We shall assume in future that $\boldsymbol{M}$ besides its objects also contains material particules of these.

In such a manner we have specified objects of the mechanical system $\boldsymbol{M}$: they are materialized mathematical objects and their homogenous parts with respect to elements of $\mathcal{M}$. In this manner, and since it is not necessary, we shall not be concerned with the structure of mathematical objects. However, this specification will be necessary for certain further purposes which are beyond the content of this paper.

Now we shall specify morphisms of $\boldsymbol{M}$. Let (G, μ) and (G', μ') be two $\mathcal{M}$-objects of $\boldsymbol{M}$. By a morphism of (G, μ) to (G', μ') we mean a morphism $\alpha: G \to G'$ such that $\alpha\mu = \mu'(\alpha \times 1_M)$. A morphism between $\mathcal{M}$-objects we shall call a *mechanical morphism*. We shall assume that *forces* being connectives between objects of $\boldsymbol{M}$ are just mechanical morphisms. The mechanical system $\boldsymbol{M}$ with so specified objects and connectives between them we shall call the fundamental mechanical world. Hence we have the following

Definition 2. By the *fundamental mechanical world* on a level in the universe $\boldsymbol{U}$ we mean a mechanical system consisting of materialized mathematical objects of the same structural type and of mechanical morphisms between them.

The above specification of the fundamental mechanical world is quite sufficient for our present purposes. However, in a forthcoming paper, we shall deal with the structure of objects of $\boldsymbol{M}$ and objects of various levels in the universe $\boldsymbol{U}$ generally, with properties of such objects, with some additional structure on the world $\boldsymbol{M}$, etc. There we shall assume that objects of the universe $\boldsymbol{U}$ are just the fundamental mechanical worlds equipped with certain additional structure which we shall call a spatial structure. This approach will require further considerations of concepts — materialization and material particules.

For further purposes, we need the concept of a morphism between fundamental mechanical worlds. Since these worlds have the structure of a fundamental semigroupoid, then the morphisms in question are fundamental homomorphisms [3]. Their definition is as follows:

Definition 3. Let $\boldsymbol{M}$ and $\boldsymbol{M}'$ be two fundamental mechanical worlds. A *fundamental homomorphism* $\boldsymbol{M}$ to $\boldsymbol{M}'$ is a triple $(\boldsymbol{M}, T, \boldsymbol{M}')$, where T is a function from the class of mechanical morphisms of $\boldsymbol{M}$ to the class of morphisms of $\boldsymbol{M}$, i.e., $T: \boldsymbol{M}_{mor} \to \boldsymbol{M}'_{mor}$, satisfying the following conditions:

1. If $\mathcal{D}_i(f) = a$, then $\mathcal{D}'_i(Tf) = Ta$, for $i = 0, 1$,
2. if $\mathcal{C}(f, g; h,)$, then $\mathcal{C}'(Tf, Tg; Th)$,

where the primes denote the unary operations $\mathcal{D}_0$ and $\mathcal{D}_1$ and the binary operation $\mathcal{C}$ on the world $\boldsymbol{M}'$.

Since we have specified fundamental mechanical worlds and morphisms between them we shall go further to specify behaviours of these worlds under the action of external and active forces. Certainly, under the action of such a force upon a mechanical world $\boldsymbol{M}$ this will move and deform. Here we shall only be concerned with the problem of motion of $\boldsymbol{M}$.

Definition 4. By the *motion* of the world $\boldsymbol{M}$ we mean a family of fundamental homomorphisms $T_t: \boldsymbol{M} \to \boldsymbol{M}$ indexed by the set of real numbers $\boldsymbol{R}$ such that $T_o \cong I$ and $T_{t'} \circ T_{t''} \cong T_{t'+t''}$ for $t', t'' \in \boldsymbol{R}$, where I is the identity homomorphism of $\boldsymbol{M}$, $\circ$ is the composition of the fundamental homomorphisms and $\cong$ denotes a natural isomorphism of fundamental homomorphisms [3].

Hence we have that $T_t, t \in \boldsymbol{R}$ is an equivalence. It means that for each T_t there is a homomorphism T_t^{-1} such that $T_t \circ T_t^{-1} \cong \mathrm{I}$ and $T_t^{-1} \circ T_t \cong I$. Thus the motion of $\boldsymbol{M}$ is a family of equivalences of it.

The motion of $\boldsymbol{M}$ is characterized by certain concepts. In what follows we shall define a necessary concept — that of an orbit. By the ***orbit*** or ***R-orbit*** of an object a of $\boldsymbol{M}$ we mean a class $O(a) = \{T_t(a) \mid t \in \boldsymbol{R}\}$. Otherwise, an object $a \in \boldsymbol{M}_{ob}$ will be ***R-invariant*** if $T_t(a) = a$ for all $t \in \boldsymbol{R}$.

By means of orbits, we can create on $\boldsymbol{M}$ a fundamental semigroupoid $O(\boldsymbol{M})$, objects of which are orbits $O(a), a \in \boldsymbol{M}_{ob}$ and morphisms, orbit-morphisms $O(f), f \in \boldsymbol{M}_{mor}$. Clearly $\boldsymbol{M}$ is a full subsemigroupoid of $O(\boldsymbol{M})$.

3. Laws of motion

In this section we shall settle the laws of motion of objects of $\boldsymbol{M}$. As we have already said, these laws correspond to those of Newton and mechanics to the Newtonian mechanics.

Certainly, a force F acting on $\boldsymbol{M}$ is a cause of the motion of $\boldsymbol{M}$ and hence of the creation of orbits within $\boldsymbol{M}$. Thus such a force F creates the semigroupoid $O(\boldsymbol{M})$ on $\boldsymbol{M}$. Moreover, it produces certain entities on objects of $\boldsymbol{M}$ and of $O(\boldsymbol{M})$ which characterize its action. Our aim is now to relate the force to these entities.

Denote by $T(a)$ the set of certain characteristics defined on an object a of $\boldsymbol{M}$ and by $T(O(a))$ the same concept for the orbit $O(a)$ of a. Within these concepts we shall find relevant ones characterizing the motion. By such a characterictic we mean a map $\rho: O(a) \to T(O(a))$ such that $\rho(a') \in T(a')$, $a' \in O(a)$.

Now we shall define one more map. It is a morphism D_t which for every $t \in \boldsymbol{R}$ assigns, to each characteristic of motion ρ, a characteristic $D_t \rho$ such that $D_t(\rho(a)) \in T(a_t)$, where a_t is an object of the orbit $O(a)$ i.e., a position of the object a in the moment t. Such a map together with the characteristic ρ will serve for the specification of the force F. Namely, they will represent the result of the action of the force.

Since a force F produces on each object a of $\boldsymbol{M}$ a characteristic ρ which changes during the motion, i.e., along the orbit of that object, then we can express it through its action, i.e., through $D_t \rho$. Thus we can postulate

$$\text{(LM)} \qquad F = D_t \rho.$$

This formula represents a *law of motion* within $\boldsymbol{M}$.

Certainly, the above law is given for objects of $\boldsymbol{M}$. However, we have to express it through laws holding for particules of these objects and moreover to give it an explicit form. We shall not here do this since we do not know yet the structure of objects of $\boldsymbol{M}$.

Nevertheless, if we assume that $\boldsymbol{M}$ is a configuration space, objects of which are its material particules, having the structure of a differentabile manifold [2], then we can obtain an explicit form of the above law:

$$(\mathrm{LM})_{\mathrm{m}} \qquad F = (\gamma^* D)_{d/du} \gamma_*,$$

where D is a connection on $\boldsymbol{M}$, γ_* is a tangent on the parametrized curve γ of $\boldsymbol{M}$; it is a characteristic entity of motion.

The expression $(LM)_m$ represents the second Newton's law in case of manifolds. From it we obtain the first law. Namely, if the action of F vanishes from a moment on, then we have

$$(\gamma^* D)_{d/du} \gamma_* = 0.$$

This formula is the differential equation of the geodesic on M. Taking this into account we can formulate the following

P r o p o s i t i o n 1. *If the force F acting on an object of the manifold* M *vanishes at a moment, then that object which has been moving will continue its moving along the geodesic.*

From the law of motion (LM), we can obtain the law of equilibrium in motion. Namely, if we add, to each external force F acting on an object of M, a force $F^* = -(D_t \rho)$, then the set of all forces $\mathcal{F}$ acting on that object will become balanced. It means that for each $F \in \mathcal{F}$ there exists a force F^* such that $F \circ F^* = e$ and $F^* \circ F = e$ is the identity force of $\mathcal{F}$. In that way we have the

P r o p o s i t i o n 2. *Any set of forces* $\mathcal{F}$ *acting on an object of* M *becomes balanced by motion.*

This is a form of the third Newton's law. According to it, for any force F acting on an object there is a force F^* which when composed with F gives as a result the identity force.

In such a manner we have specified a part of mechanics which corresponds to Newtonian mechanics. Our next task will be an extension of these considerations including deformations of mechanical objects. In that way we shall contribute to the specification of the universe U. However, a complete specification of U will require a certain further work.

4. Conclusion

This paper represents an attempt to consider some fundamental questions of mechanics. This attempt may serve as a base for the rigorous — the axiomatic approach to this theory. If one decides to such an approach, then he has to start from the concepts which are pointed out as primititive. Thus, one might say that the primitive concepts of mechanics are those of mathematics to which certain mechanical — material concepts are added, as that of mass for instance. The axioms of this theory are those which specify the mathematical theory in question and those which connect mathematical and mechanical concepts. Our further investigations will contribute to this problem.

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