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ON TOPOLOGICAL ERGODIC MEASURE SPACES

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§ 1. Introduction

In this paper we have made some investigations on topological ergodic measure spaces, i.e., topological measure spaces having a nonvoid collection of ergodic autohomeomorphisms. The notion of almost-tractable space has been introduced in [2] and in fact the notion of topological ergodic measure space is a realisation of the notion of almost-tractable space. In the first theorem it has been shown that the topological space associated with a topological ergodic measure space is almost-tractable. Further, in the present article, we have grouped topological ergodic measure spaces into two classes viz., balanced and unbalanced topological ergodic measure spaces. It has been observed that if a topological ergodic measure space is balanced and a set of second category, then the topology associated with the space has unusual characteristics like that of a regular Baire Category measure space [See 5]. Further, if the σ-algebra Σ of a balanced Baire finite topological ergodic measure space (X, J, Σ, μ) is a Borel σ-algebra and μ is complete, then the space (X, J, Σ, μ) is a category space. Along with these we have also investigated some other characterictic properties of such spaces and proved some theorems dealing with the conditions under which topological ergodic measure spaces are balanced and unbalanced. All the topological spaces considered in this paper are T_1-spaces and the continuum hypothesis will be assumed when required.

A topological measure space (X, J, Σ, μ) is a measure space (X, Σ, μ) with a topology J such that $J \subseteq \Sigma$ and $\mu(\mathrm{U}) > 0$ for every nonvoid U in J. (X, J, Σ, μ) is said to be a finite topological measure space if μ is finite. If x be a point and $A\,(x \notin A)$ be a subset of (X, J, Σ, μ) such that $\mu(\{x\}) = \mu(A) = 0$ and if there exist open sets U_1 and U_2 such that $\{x\} \subseteq U_1$, but $A \cap U_1 = \varnothing$ and $A \subseteq U_2$, but $\{x\} \cap U_2 = \varnothing$, then the space (X, J, Σ, μ) is said to be an 0-separated topological measure space. A topological space is said to be a quasi-regular if every nonvoid open set contains a nonvoid regularly closed set. A topological space is said to be totally nonmeagre if every closed set is, as a subspace, a Baire Space. The least cardinal number of a nonvoid open set of a topological space is said to be the dispersion character of the space and it is denoted by η. A set V of a topological space (X, J) is said to be an α-set of the space if Int Cl Int $V \supseteq V$, where Int and Cl denotes the interior and closure operators of the space (X, J). The class of α-sets of (X, J) is denoted by J_α. The topology J of the space (X, J) is said to be an α-topology if $J = J_\alpha$. A topological space (X, J) is said to be a tractable space if for every nonvoid proper closed subset K of the space there exists an autohomeomorphism g of the space such that $g(K) \neq K$. A topological space (X, J) is said to be an almost-tractable space if for every nonvoid proper regularly closed set K' there exists an autohomeomorhism g' of the space such that $g'(K') = K'$. Let H be a group of autohomeomorphisms of a space (X, J) and x be a point. The set of all the images of x under the members of H is said to be an orbit of the space generated by x under H or simply an orbit of the space under H, when there is no chance of confusion.

§ 2. Definition

A topological space measure space (X, J, Σ, μ) is said to be a topological ergodic measure space if the class of ergodic (see [3]) autohomeomorphism Δ of the space (X, J, Σ, μ) is nonempty.

Let (X, J, Σ, μ) be a topological ergodic measure space and let $g \in \Delta$. Let $\mathcal{S}_g$ be the collection of all the invariant open sets under C_g, where C_g is the cyclic group of autohomeomorphisms of the space generated by g. Let S_g denote the intersection of all the members of $\mathcal{S}_g$, then S_g is said to be the K-set of the space generated by C_g or simply a K-set of the space, when there is no chance of confusion.

Definition: If every dense measurable subset of a topological ergodic measure space (X, J, Σ, μ) is of positive measure and μ is nonatomic (See [5]), then the space (X, J, Σ, μ) is said to be a balanced space; otherwise it is said to be an unbalanced space.

Theorem 1. *If (X, J, Σ, μ) be a topological ergodic measure space, then the space (X, J) is an almost-tractable space.*

Proof. Let K be a nonvoid proper regularly closed subset of the space (X, J, Σ, μ). As Int $K \neq \varnothing$, we have $\mu(K) > 0$. Also, it is obvious that $\mu(CK) > 0$. Therefore, for any $g \in \Delta$ $g(K) \neq K$. It follows that the space (X, J) is an almost-tractable space.

Corollary 1. *If (X, J, Σ, μ) be a topological ergodic measure space with X as a set of second category, then the space (X, J) is a Baire Space.*

Proof. As the space (X, J) is almost-tractable it follows from the Theorem 2 of [2] that the space (X, J) is a Baire Space.

Theorem 2. *If (X, J, Σ, μ) be a balanced topological ergodic measure space, then $\omega(X) \geqslant c$, where $\omega(X)$ is the weight of the space (X, J).*

Proof. If it is assumed that $\omega(X) \ngeqslant c$, then the assumption of the continuum hypothesis implies that the space (X, J) satisfies the second axiom of countability. Let $\mathcal{B}$ be a countable base of the space (X, J) and let $\mathcal{S}_g'$ be the collection of all the nonvoid invariant open sets of the space generated by the members of $\mathcal{B}$ under C_g, where $g \in \Delta$. Let S_g' be the intersection of all the members of $\mathcal{S}_g'$. Then it is obvious that S_g' is the K-set of the space generated by C_g. Since every member of $\mathcal{S}_g'$ is a dense invariant open subset of (X, J) under C_g, the complement of every member of $\mathcal{S}_g'$ is a null set. Also the collection $\mathcal{S}_g'$ is countable. Hence we have $\mu(S_g')=\mu(X)$. Further, as S_g' is a K-set of the space generated by C_g, every nonvoid open set invariant under C_g contains S_g'. It follows that every orbit of the space (X, J) under C_g contained in S_g' is a dense subset of (X, J). Let $A \subseteq S_g'$, where A is an orbit of the space under C_g. Obviously A is countable and measurable. Now, if it is assumed that $\mu(A)>0$, then μ is atomic and so the space (X, J, Σ, μ) is not balanced. Again as A is a dense measurable subset of (X, J, Σ, μ) and the space is balanced, we have $\mu(A)>0$ and consequently u is atomic, a contradiction. Hence it follows that $\omega(X) \geqslant c$.

Theorem 3. *If (X, J, Σ, μ) be balanced topological ergodic measure space such that X is a set of second category, then a measurable set is a null set if and only if it is a set of first category.*

Proof. Let A be a measurable set of first category and let us assume that $\mu(A)>0$. Let $A^* = \bigcup_{n \in N^*} g^n(A)$, where N^* is the set of integers and $g \in \Delta$. Then obviously A^* is a set of positive measure invariant under g and consequently $\mu(CA^*)=0$. Now, as the space (X, J, Σ, μ) is balanced and $\mu(CA^*)=0$, we have $\operatorname{Int} A^* \neq \varnothing$. It implies that the space (X, J) contains a nonempty open set of first category. Hence it follows that the space (X, J) is not a Baire space. But as X is a set of second category, it follows from the Corollary 1 of Theorem 1 that the space (X, J) is a Baire space, a contradiction. Hence it follows that every measurable set of first category is a null set.

To establish the converse result, let us assume that there exists a set of second category D such that $\mu(D)=0$. Obviously we have $\operatorname{Int} \operatorname{Cl} D \neq \varnothing$. Let $D^* = \bigcup_{n \in N^*} g'^n(D)$, where $g' \in \Delta$ and let us write $K = \operatorname{Cl} D$. Now, D is obviously a dense subset of the subspace (K, J_k) of (X, J). Let $K^* = \bigcup_{n \in N^*} g'^n(K)$. As $\mu(K)>0$, K^* is obviously a dense subset of (X, J). It follows that D^* is also a dense subset of (X, J). Further, we have $\mu(D^*)=0$ and consequently the space (X, J, Σ, μ) is unbalanced, a contradiction. Hence $\mu(D)>0$.

Corollary 1. *If (X, J, Σ, μ) be a balanced topological ergodic measure space such that X is a set of second category, then the space (X, J, Σ, μ) is O-separated if and only if J is an α-topology.*

Proof. In order to prove the corollary we first show that under the hypothesis of the corollary every set of first category is nowhere dense. Let us assume that there exists a set of first category A which is not a nowhere dence subset of (X, J). Then clearly we have $\operatorname{Int} \operatorname{Cl} A \neq \varnothing$. Let $A^* = \bigcup_{n \in N^*} g^n(A)$, where $g \in \Delta$. Obviously A is a dense subset of (X, J) and also it is clear from the above theorem that A^* is contained in a set of first category A^{**} such that $\mu(A^{**})=0$. It follows that the space (X, J, Σ, μ) is unbalanced, a contradiction. Hence A is nowhere dense. Now, if (X, J, Σ, μ) is O-separated, then as μ is nonatomic and every measurable nowhere dense set is a null set, it can be easily shown that every nowhere dense set is closed. Consequently, it follows from the corollary of Proposition 4 of [4] that J is an α-topology.

The converse result can be easily established.

The corollary 1 of Theorem 3 shows that in a balanced topological ergodic measure space which is also a set of second category every set of first category is nowhere dense. Therefore, it follows that the togology associated with such a topological ergodic measure space is of unusual characteristics and the Theorem 22.2 of [5] shows that such a topology resembles the topology associated with a regular Baire category measure space. It is to be further stated that if the σ-algebra of a balanced Baire finite topological ergodic measure space (X, J, Σ, μ) is a Borel σ-algebra and μ is complete, then Σ has the Baire property and the space (X, J, Σ, μ) is a category measure space.

Definition: A set of positive measure A of a topological ergodic measure space (X, J, Σ, μ) is said to be a ν-set, if for every subset of positive A' contained in A Int Cl $A' \supseteq A'$.

Theorem 4. *If (X, J, Σ, μ) be a balanced topological ergodic measure space, then the space (X, J) is a Baire space if and only if $(X, J. \Sigma, \mu)$ contains a ν-set.*

Proof. Let A be ν-set of the space (X, J, Σ, μ). If it is assumed that (X, J) is not a Baire space, then there exists a nowhere dense subset $\mathscr{B}$ such that $\mu(B)>0$. Let $g \in \Delta$. Then clearly, as g is ergodic, there exists an integer n such that $\mu(A \cap g^n(B))>0$. Further, as g is an autohomeomorphism of the space (X, J, Σ, μ), $A \cap g^n(B)$ is nowhere dense. It follows that A is not a ν-set of the space, a contradiction. Hence it follows that the space (X, J) is a Baire space.

To establish the converse result, let us assume that the space (X, J) is a Baire space. If it is assumed that the space (X, J, Σ, μ) contains no ν-set, then obviously there exists a nowhere dense subset B' which is of positive measure. Let $B^* = \bigcup_{n \in N'} g'^n(B')$, where $g' \in \Delta$. As $\mu(CB^*)=0$ and (X, J, Σ, μ) is balanced, we have Ind $B^* \neq \varnothing$. It follows that Int B^* is a set of first category and consequently it follows that (X, J) is not a Baire space, a contradiction. Hence it follows that the space (X, J, Σ, μ) contains a ν-set and the proof of the corollary is complete.

Theorem 5. *Let (X, J, Σ, μ) be a topological ergodic measure space containing a ν-set and having no isolated point. If every null set of the space (X, J, Σ, μ) is contained in a G_δ null set, then the space (X, J, Σ, μ) is balanced.*

Proof. In order to show that the space (X, J, Σ, μ) is balanced we are to show that μ is nonatomic and the space (X, J, Σ, μ) contains no dense null set. If it is assumed that μ is atomic, then as the space contains a ν-set it can be easily shown that the class of isolated points of the space is nonvoid. By hypothesis, the space has no isolated points and hence μ is nonatomic. Again, if it is assumed that the space contains a dense null set A, then there exists a G_δ null set A^* such that $A \subseteq A^*$. It follows that $\mu(CA^*)>0$ and also CA^* is a set of first category. Therefore, there exists a nowhere dense set $A' \subseteq CA^*$ such that $\mu(A')>0$. But, as by hypothesis, the space (X, J, Σ, μ) contains a ν-set it can be easily shown following the arguments of the Theorem 4 that every set of positive measure is a ν-set and consequently A' cannot be a nowhere dense set, a contradiction. Hence it follows that the space (X, J, Σ, μ) is balanced.

Theorem 6. *If* (X, J, Σ, μ) *be an* O*-separated topological ergodic measure space such that* $\eta > \aleph_0$, *then the space* (X, J, Σ, μ) *is a balanced space.*

Proof. To establish the theorem we first show that under the hypothesis of the theorem μ is nonatomic. Let $g \in \Delta$. Now, if it is assumed that μ is atomic, then obviously the K-set $S_g \neq \varnothing$. Under the situation, S_g cannot consist of more than one orbit of the space under C_g. For, if it is assumed that the class of orbits of the space under C_g contained in S_g is nondegenerate, then every orbit excepting one is a null set. Let A^* be an orbit of the space under C_g such that $A^* \subseteq S_g$ and $\mu(A^*) = 0$. Obviously A^* is infinite. Let the point $x \in A^*$. Then as $\mu(A^* - \{x\}) = \mu\{x\} = 0$ and the space (X, J, Σ, μ) is O-separated, there exists two open sets U_1 and U_2 such that $(A^* - \{x\}) \subseteq U_1$, $\{x\} \cap U_1 = \varnothing$ and $\{x\} \subseteq U_2$, but $(A^* - \{x\}) \cap U_2 = \varnothing$. Let us write $U' = U_2 \cap C\{x\}$. As the dispersion character of the space (X, J) is infinite $U' \neq \varnothing$. Also, we have $A^* \cap U' = \varnothing$. But, as $A^* \subseteq S_g$ it is a dense subset of (X, J) and therefore $A^* \cap U' \neq \varnothing$, a contradiction. Consequently, S_g consists of one orbit of the space under C_g. Now, as $\mu(S_g) > 0$, we have $\mu(CS_g) = 0$. Further, as the space (X, J, Σ, μ) is O-separated, it can be easily shown by arguments similar to above that S_g is open. It contradicts the hypothesis $\eta > \aleph_0$ and consequently it follows that μ is nonatomic. Now, it can be easily shown that every null set is closed and therefore every dense measurable set is of positive measure and consequently the space (X, J, Σ, μ) is balanced.

Theorem 7. *Let* (X, J, Σ, μ) *be a finite topological ergodic measure space with* μ *complete and* card X *of measure zero and let* A *be a subset of the space such that* μ $(\mathrm{Cl}\, A) > 0$ *and every subset of* A *is measurable. Then the space* (X, J, Σ, μ) *is unbalanced.*

Proof. If μ is atomic, then the theorem is obviously true. Let us consider the case when μ is nonatomic. Let $A^* = \bigcup_{n \in N^*} g^n(A)$, where $g \in \Delta$. Since $\mu(\mathrm{Cl}\, A) > 0$, it is obvious that A^* is a dense subset of (X, J). Now, we show that $\mu(A^*) = 0$. If it is assumed that $\mu(A^*) > 0$, than clearly we have $\mu(CA^*) = 0$. As μ is complete every subset of CA^* is a null set and also, as every subset of A is measurable, every subset of A^* is measurable. It follows that Σ is the power set of X. By hypothesis μ is finite and card X is of measure zero. Hence it follows that μ is atomic, a contradiction. Therefore, $\mu(A^*) = 0$ and A^* is a dense null set and hence it follows that the space (X, J, Σ, μ) is unbalanced.

Corollary 1. *Let* (X, J, Σ, μ) *be a finite topological ergodic measure space such that* J *is an* α*-topology and* card X *of measure zero and let* μ *be complete and nonatomic. Then the space* (X, J, Σ, μ) *is totally nonmeagre.*

Proof. We first show that (X, J) is a Baire space. If it is assumed that (X, J) is not a Baire space, then it follows the Corollary 1 of Theorem 1 that X is a set of first category and hence (X, J) contains a nowhere dense set A of positive measure. Now, as J is an α-topology, it follows from the corollary of the proposition 4 of [4] that every subset of A is closed and hence measurable. As, by hypothesis, μ is finite and complete and card X is of measurable zero it can be be easily shown by arguments similar to that of Theorem 7 that μ is atomic. But, by hypothesis, μ is nonatomic and consequently it follows that the space (X, J) is a Baire space. Now, if a closed subset K

of (X, J) has void interior, then as J is an α-topology K is, as a subspace, a discrete space and so it is, as a subspace, a Baire space. On the other hand, a closed subset of (X, J) with nonvoid interior is obviously, as a subspace, is a Baire space and hence it follows that (X, J) is totaly nonmeagre.

Theorem 8. *If* (X, J, Σ, μ) *be a topological ergodic measure space and* $A \in \Sigma$ *such that* $\mu(\mathrm{Cl}\, A) > 0$ *and* $\mathrm{card}\, A < \eta$, *then the space* (X, J, Σ, μ) *is balanced.*

Proof. Let us assume that (X, J, Σ, μ) is balanced and let A^* be the union of all the images of A under C_g, where $g \in \Delta$. Since $\mu(\mathrm{Cl}\, A) > 0$, it is obvious that A^* is a dense subset of (X, J). As the space (X, J, Σ, μ) is balanced and A^* is a dense measurable set, we have $\mu(A^*) > 0$. Also, we have $g(A^*) = A^*$. Therefore, $\mu(CA^*) = 0$ and CA^* is nondense. It follows that Int $A^* \neq \varnothing$. Further, as the space (X, J, Σ, μ) is balanced, μ is nonatomic, and consequently card $A^* > \aleph_0$. Hence we have card $A =$ card A^*. As Int $A^* \neq \Phi$, we have $\eta \leq$ card $A^* =$ card A. But, by hypothesis we have card $A < \eta$, a contradiction. It follows that the space (X, J, Σ, μ) is unbalanced.

Definition. A decomposition $\mathcal{D}$ of a topological ergodic measure space (X, J, Σ, μ) is said to be nontrivial at a point x if for $x \in A \subseteq U$, where $A \in \mathcal{D}$ and $U \in J$, there exists an open set U' such that $x \in A \subseteq U' \subseteq U$ and the union of all the members $\mathcal{D}$ contained in U' is a set of positive measure. If $\mathcal{D}$ is a decomposition non-trivial at every point, then it is said to be a non-trivial decomposition.

If f be an autohomeorphism of a topological ergodic measure space (X, J, Σ, μ), then the class of orbits of the space under C_f, where C_f is the cyclic group of autohomeomorphisms of the space generated by f, forms a decomposition of the space. In the following such a decomposition will be denoted by $\mathcal{D}_f$.

Theorem 9. *If* (X, J, Σ, μ) *be a topological ergodic measure space with* (X, J) *as a quasi-regular space and if there exists an element* $g \in \Delta$ *such that the decomposition* $\mathcal{D}_g$ *is nontrivial at a point* x, *then the space* (X, J, Σ, μ) *is unbalanced.*

Proof. If μ is atomic, then the theorem is obviosly true. Let μ be nonatomic and $x \in A$, where $A \in \mathcal{D}_g$. Now, we show that under the hypothesis of the theorem A is a dense subset of the space. Let us assume that A is nondense. Then $\mathrm{Cl}\, A \neq X$ and as the space (X, J) is quasi-regular there exists a proper regularly open set U such that $\mathrm{Cl}\, A \subseteq 0$. As the decomposition $\mathcal{D}_g$ is nontrivial at the point x, there exists an open set U' such that $x \in A \subseteq U' \subseteq U$ and the union A^* of all the members of $\mathcal{D}_g$ contained in U' is a set of positive measure. It is obvious that Int $C\,A^* \neq \varnothing$. Therefore, $\mu(CA^*) > 0$ and consepuently we have, $g(A^*) \neq A^*$. But as A^* is the union of a collection of orbits of the space under C_g, we have $g(A^*) = A^*$, a contradiction. It follows that A is dense in (X, J). Further, as μ is nonatomic and card $A = \aleph_0$, we have $\mu(A) = 0$. Hence it follows that the space (X, J, Σ, μ) is unbalanced.

Corollary 1. *If* (X, J, Σ, μ) *be a topological ergodic measure space with* (X, J) *quasi-regular and if there exists an element* $g' \in \Delta$ *such that the decomposition* $\mathcal{D}_{g'}$ *is nontrivial, then the space* (X, J) *is a separable tractable space.*

Proof. The separability of the space (X, J) is obvious from the above theorem. Also as the decomposition $\mathcal{D}_{g'}$ is nontrivial, it follows from the Theorem 9 that every member of $\mathcal{D}_{g'}$ is a dense subset of (X, J). Again, $\Delta \subseteq G$, where G is the group of all the autohomeorphisms of the space (X, J). Therefore, every member of $\mathcal{D}_{g'}$ is contained in some orbit of the space (X, J) under G. It follows that every collection of the space (X, J) (See [1]) is a cover of the space (X, J) and hence it follows from the Theorem 1 of [1] that the space (X, J) is tractable.

In the following we cite some examples of topological ergodic measure spaces with certain characteristic properties.

Example 1. Let (X, J) be the subspace of the real line with $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ and J as the usual subspace topology. Let Σ be the smallest σ-algebra of subsets of X containing the members of J. Let μ be a measure on Σ such that $\mu\left(\left\{\frac{1}{n}\right\}\right) = 1$ for every $n \in N$, where N is the set of positive integers and let $\mu(\{0\}) = 0$. Obviously (X, J, Σ, μ) is a topological measure space with μ atomic. Let g be a one-one correspondence of (X, J) onto itself such that 0 is a fixed point of g. $g\left(\frac{1}{n}\right) = \frac{1}{2+n}$, when n is even. $g\left(\frac{1}{n}\right) = \frac{1}{n-2}$, when n is odd and greater than 1 and $g(1) = \frac{1}{2}$. Now, from the construction of g it is obvious that if A is a set of positive measure invariant under g, then either $A = X - \{0\}$ or $A = X$. Also g is measurable and non-singular. Hence g is a ergodic transformation. Further, as (X, J) is a metric space and the image and inverse image of every convergent sequence in (X, J) are convergent sequences, g is an autohomeomorphism of the space. Again (X, J, Σ, μ) is obviously O-separated and unbalanced. If K be a nonvoid proper regularly closed subset of the space, then from the construction of the mapping g it is obvious that $g(K) \neq K$. Hence it follows that the space (X, J) is almost-tractable.

Example 2. Let X be a set such that $\operatorname{card} X = C$ and let X be associated with co-countable topology J. Let A_1 and A_2 be two mutually exclusive subset of X such that $A_1 \cup A_2 = X$ and $\operatorname{card} A_1 = \operatorname{card} A_2 = \operatorname{card} X$. Let Σ be the smallest σ-algebra of subsets of X containing the collection of sets $\{A_1, A_2\}$ and J. Let μ be a finite measure on Σ such that a set is of measure zero if and only if it is a set of first category. Obviously (X, J, Σ, μ) is a topological measure space. Let g be a one-one correspondence of the space (X, J, Σ, μ) onto itself such that $g(A_1) = A_2$, $g(A_2) = A_1$. Then clearly g is an ergodic autohomeomorphism of the space (X, J, Σ, μ). Hence $\Delta \neq \varnothing$ and the space (X, J, Σ, μ) is a topological ergodic measure space. Further, it is obvious that every dense measurable subset of the space is a set of positive measure and also μ is non-atomic. Therefore, it follows that the space (X, J, Σ, μ) is balanced. Further, the space (X, J) is obviously T_1 and totally nonmeagre. Also the topology of the space (X, J) is an α-topology and its dispersion character $\eta > \aleph_0$. It can be easily shown that (X, J, Σ, μ) is O-separated and μ is complete.

Example 3: Let (X^*, J^*) be a topological space with card $X^* = c$ and J^* be a co-finite topology. Let $A_1^* \cup A_2^* = X^*$, $A_1^* \cap A_2^* = \varnothing$ and card $A_1^* =$

= card A_2^* = card X^*. Let Σ^* be the smallest σ-algebra containing the collections of sets $\{A_1^*, A_2^*\}$ and J^*. Let μ^* be a measure defined on Σ^* such that a set in Σ^* is of measure zero if and only if it is a set of first category. Then obviously $(X^*, J^*, \Sigma^*, \mu^*)$ is a topological ergodic measure space. It can be easily shown that the space $(X^*, J^*, \Sigma^*, \mu^*)$ is unbalanced. Further, clearly μ^* is nonatomic and complete.

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