PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 21 (35) 1977, pp. 5–7

NON-UNIQUE FIXED POINTS IN L-SPACES

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(Received December 15, 1976)

Introduction

In a recent paper Ćirić [1] proved some fixed point theorem when the mapping T of a metric space (M, d) satisfies the following inequality

(1) $\min\{d(Tx, Ty), d(x, Tx), (y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\}) \leqslant \alpha\, d(x, y)$

for $x, y \in M$ and for some $0 < \alpha < 1$.

The chief aim of the present paper is to study the above mapping in spaces of type L of Fréchet, which we shall call separated L-spaces. A similar result in non-separated L-spaces will also be stated.

1. Some Definitions

Definition 1. Let N denote the set of all non-negative integers. A pair $(M, \rightarrow)$ of a set M and a subset $\rightarrow$ of the set $M^N \times M$ is called an L-space if the following conditions are satisfied:

(L-1) If $x_n = x \in M$ for all $n \in N$, then $(\{x_n\}_{n \in N}, x) \in \rightarrow$

(L-2 If $(\{x_n\}_{n \in N}, x) \in \rightarrow$, then $(\{x_{n_i}\}_{i \in N}, x) \in \rightarrow$

for every subsequence $\{x_{n_i}\}_{i \in N}$ of $\{x_n\}_{n \in N}$. In what follows, we shall write $\{x_n\}_{n \in N} \rightarrow x$ or $x_n \rightarrow x$ instead of $(\{x_n\}_{n \in N}, x) \in \rightarrow$, and read $\{x_n\}_{n \in N}$ converges to x.

Definition 2. Let $(M, \rightarrow)$ be an L-space. It is said to be separated if each sequence in M converges to atmost one point of M.

Definition 3. Let d be a non-negative extended real valued function on $M \times M$: $0 \leqslant d(x, y) \leqslant \infty$ for all $x, y \in M$. The L-space $(M, \rightarrow)$ is said to be d-complete if each sequence $\{x_n\}_{n \in N}$ in M with $\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$ converges to atmost one point of M.

2. Main theorems

Theorem 1. *Let* $(M, \to)$ *be a separated L-space which is d-complete for a non-negative extended real valued function d on* $M \times M$ *and T be a continuous mapping of M into itself satisfying the following conditions for some* α, β *with* $0<\alpha<1, 0<\beta\leqslant\infty$:

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \leqslant \alpha\, d(x, y) \tag{2}$$

for $x, y \in M$ *with* $d(x, y) < \beta$

$$d(Tb, b) < \beta \text{ for some } b \in M. \tag{3}$$

Then T has a fixed point and the sequence $\{T^n b\}_{n\in N}$ *converges to the fixed point.*

Proof. Now

$$\min\{d(T^{n+1}b, T^n b), d(T^n b, T^{n+1}b), d(T^n b, T^{n-1}b)\} - \min\{d(T^{n-1}b, T^{n+1}b),$$

$$d(T^n b, T^n b)\} \leqslant \alpha\, d(T^{n-1}b, T^{n+1}b)$$

i.e.

$$d(T^{n+1}b, T^n b) \leqslant \alpha\, d(T^{n-1}b, T^n b)$$

By induction we get

$$d(T^{n+1}b, T^n b) \leqslant \alpha^n d(Tb, b)$$

for every $n \in N$ and so we have $\sum_{n=0}^{\infty} d(T^{n+1}b, T^n b) < \infty$. Hence the d-completeness of the space implies that the sequence $\{T^n b\}_{n\in N}$ converges to some $u \in M$. So, by the continuity of T, there is a subsequence $\{T^{n_i} b\}_{i\in N}$ of $\{T^n b\}_{n\in N}$ such that $T(T^{n_i} b) \to Tu$. But then since $\{T(T^{n_i} b)\}_{i\in N}$ is a subsequence of $\{T^n b\}_{n\in N}$, we have $T(T^{n_i} b) \to u$. Therefore $Tu = u$. This completes the proof of the Theorem.

Theorem 2. *Let* $(M, \to)$ *be an L-space which is d-complete for a continuous non-negative extended real valued function d on the product space* $M \times M$ *with the property that* $d(x, y) = 0$ *implies* $x = y$. *If T be a continuous mapping of M into itself satisfying conditions* (2) *and* (3) *of Theorem* 1 *for some* α, β *with* $0<\alpha<1$, $0<\beta\leqslant\infty$, *then T has a fixed point.*

Proof. By induction (as in theorem 1)

$$d(T^{n+1}b, T^n b) \leqslant \alpha^n d(Tb, b) \tag{4}$$

for every $n \in N$. Hence she same argument employed in the proof of Theorem 1 yields that the sequence $\{T^n b\}_{n\in N}$ converges to some $u \in M$ and that $T(T^{n(k)} b) \to Tu$ for some subsequence $\{T^{n(k)} b\}$ of the sequence $\{T^n b\}_{n\in N}$. Therefore the continuity of T implies that

$$d(T(T^{n(k_i)} b), T^{n(k_i)} b) \to d(u, Tu)$$

for some subsequence $\{T^{n(k_i)}\}_{i\in N}$ of $\{T^{n(k)}b\}_{k\in N}$. However (4) shows that

$$d(T(T^{n(k_i)}b), T^{n(k_i)}b)\to 0.$$

Hence $d(u, Tu)=0$ and thus we have $Tu=u$.

Acknowledgement. The author expresses his sincere thanks to Prof. S. Kasahara for his valuable support.

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