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ON SOME FUNCTIONAL-DIFFERENTIAL EQUATIONS

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1. In the present paper we consider the sequences of following functional-differential equations

(1) $$y'(x)=F_n(x,\ y(x),\ y[f(x)]), \qquad n=0,\ 1,\ \ldots$$

and

(2) $$y'(x)=F_n(x,\ y(x),\ y[f_n(x)]), \qquad n=0,\ 1,\ \ldots$$

We shall prove some theorems about the existence, uniqueness and continuous dependence on the given functions of solutions of these equations. The corresponding problem for the equations

(3) $$y'(x)=F_n(x,\ y(x)), \qquad n=0,\ 1,\ \ldots$$

has been investigated in [2].

2. Let R be the space of real numbers and let

(4) $$Y=\{(x,\ y,\ z):|\,x-x_0\,|\leqslant a,\ |\,y-y_0\,|\leqslant b,\ |\,z-z_0\,|\leqslant b,\ a,\ b>0\},$$

$$x,\ y,\ z,\ x_0,\ y_0\ z_0,\ a,\ b\in R.$$

We assume the following hypotheses:

(i) The functions $F_n: Y\to R$, $n=0,\ 1,\ \ldots,$ are continuous on Y. There exist positive number M and bounded sequences of positive numbers $\{L_n\}$, $\{K_n\}$, $n=0,\ 1,\ \ldots,$ such that

(5) $$|F_n(x,\ y,\ z)|\leqslant M \text{ for all } (x,\ y,\ z)\in Y, \qquad n=0,\ 1,\ \ldots$$

and

(6) $$|F_n(x,\ y,\ z)-F_n(x,\ y_1,\ z_1,)|\leqslant L_n|\,y_1-y\,|+K_n|\,z_1-z\,|$$
$$\text{for all } (x,\ y,\ z),\ (x,\ y_1,\ z_1)\in Y, \qquad n=0,\ 1,\ \ldots.$$

(ii) The function $f: I\to R$ is continuous on I and $f(x)\leqslant x$ for $x\in I$, where $I=\langle x_0-a,\ x_0+a\rangle$.

(iii) The sequence $\{F_n\}_{n=1}^{\infty}$ converges pointwise on Y to F_0.

The following result is known.

Theorem 1. (Nadler [2]). *Let (X, d) be a locally compact metric space, let $T_n: X \to X$ be a contraction mapping with fixed point a_n for $n=0, 1, \dots$ If the sequence $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T_0, then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a_0.*

Now we shall prove

Theorem 2. *Let hypoteses* (i), (ii), (iii) *be fulfilled. Then there exists $0 < h \leq a$ such that on the interval $J = \langle x_0 - h, x_0 + h \rangle$ the initial value problem*

$$(7)\qquad \begin{aligned} & y'(x) = F_n(x, y(x), y[f(x)]), && x \in J \\ & y(x) = \varphi_0(x), && x \in \langle \alpha, x_0 - h \rangle \end{aligned}$$

where $\alpha = \min_{x \in J} f(x)$ and $\varphi_0: \langle \alpha, x_0 - h \rangle \to \langle y_0 - b, y_0 + b \rangle$ is given continuous function, has for every $n = 0, 1, \dots$ exactly one solution y_n given as the limit of successive approximations. Moreover, the sequence $\{y_n\}_{n=1}^{\infty}$ converges uniformly on J to y_0.

Remark. If $\alpha > x_0 - h$, we assume $\langle \alpha, x_0 - h \rangle = \{x_0 - h\}$.

Proof. From the boundedness of the sequences $\{L_n\}_{n=0}^{\infty}$ and $\{K_n\}_{n=0}^{\infty}$ there exists positive number $h \leq a$ such that

$$(8)\qquad 0 < 2h(L_n + K_n) < 1, \qquad n = 0, 1, \dots$$

and such that $Mh \leq b$. Let G be a space of these functions $\varphi: J \to \langle y_0 - b, y_0 + b \rangle$ which fulfil a Lipschitz condition with Lipschitz constant less than or equal to M and such that $\varphi(x_0 - h) = \varphi_0(x_0 - h)$. Let X be a space of all functions ψ defined as follows

$$\psi(x) = \begin{cases} \varphi(x), & x \in J, \\ \varphi_0(x), & x \in \langle \alpha, x_0 - h \rangle, \end{cases}$$

where $\varphi \in G$. Then X and G with the supremum metric d are compact metric spaces.

The problem (7) is equivalent with the equation

$$\begin{aligned} & y(x) = \varphi_0(x_0 - h) + \int_{x_0 - h}^{x} F_n(s, y(s), y[f(s)])\, ds, && x \in J, \\ & y(x) = \varphi_0(x), && x \in \langle \alpha, x_0 - h \rangle. \end{aligned}$$

Now for each $n = 0, 1, \dots$, and $\psi \in X$, we define $T_n(\psi)$ at each $x \in J \cup \langle \alpha, x_0 - h \rangle$ by the formula

$$(9)\qquad \begin{aligned} & T_n(\psi)(x) = \varphi_0(x_0 - h) + \int_{x_0 - h}^{x} F_n(s, \psi(s), \psi(f(s)))\, ds, && x \in J, \\ & T_n(\psi)(x) = \varphi_0(x), && x \in \langle \alpha, x_0 - h \rangle. \end{aligned}$$

From (i), (ii) it is easy to verify that for each $n=0, 1, \ldots, T_n$ maps X into itself. We also have from (9), (i) for $\psi_1, \psi_2 \in X$ and $n=0, 1, \ldots$

$$d[T_n(\psi_1), T_n(\psi_2)] =$$

$$= \sup_{x \in J \cup \langle \alpha, x_0-h\rangle} \left| \int_{x_0-h}^{x} [F_n(s, \psi_1(s), \psi_1[f(s)]) - F_n(s, \psi_2(s), \psi_2[f(s)])] \, ds \right|$$

$$\leqslant 2h(L_n+K_n) \sup_{x \in J} |\psi_1(x) - \psi_2(x)| = 2h(L_n+K_n)\, d(\psi_1, \psi_2)$$

which means, in view of (8), that T_n, $n=0, 1, \ldots$, is a contraction mapping from X into X with Lipschitz constant $2h(L_n+K_n)<1$.

From the known Banach's fixed-point principle for contraction maps [1] there exists exactly one fixed point $\psi \in X$ of transformation (9), i.e. exactly one solution $\psi \in X$ of equation (7) given as the limit of successive approximations.

Now we shall prove that $\psi_n \to \psi_0$, uniformly on J as $n \to \infty$. From (i), (iii) we have $F_n \to F_0$ pointwise and $|F_n| \leqslant M$ for $n=1, 2, \ldots$ Consequently the Lebesgue bounded convergence theorem ([3], p. 295) implies that

$$\int_{x_0-h}^{x} F_n(s, \psi(s), \psi[f(s)]) \, ds \to \int_{x_0-h}^{x} F_0(s, \psi(s), \psi[f(s)]) \, ds \qquad \text{as} \quad n \to \infty,$$

hence $\{T_n(\psi)\}_{n=1}^{\infty}$ converges pointwise on $J \cup \langle \alpha, x_0-h\rangle$ to T_0. We also have

$$|T_n(\psi)(x) - T_n(\psi)(z)| \leqslant M|x-z|, \qquad x, z \in J \cup \langle \alpha, x_0-h\rangle,$$

which means that the sequence $\{T_n(\psi)\}_{n=1}^{\infty}$ is equicontinuity on the compact set $J \cup \langle \alpha, x_0-h\rangle$. This implies (see [3], p. 162) that the sequence $\{T_n(\psi)\}_{n=1}^{\infty}$ converges uniformly on $J \cup \langle \alpha, x_0-h\rangle$ to T_0 and hence the sequence $\{T_n\}_{n=1}^{\infty}$ converges pointwise on X to T_0. From Theorem 1, the sequence of unique fixed points of transformations T_n for $n=1, 2, \ldots$, tends to the unique fixed point of T_0. Since these fixed points are equal the unique solutions of the equations (7), we have $\psi_n \to \psi_0$ uniformly on $J \cup \langle \alpha, x_0-h\rangle$, which completes the proof.

We shall assume that

(iv) The function $f: I \to R$ is continuous on I and $f(x) \geqslant x$ for $x \in I$.

We have

Theorem 3. *Let hypotheses* (i), (iii), (iv) *be fulfilled. Then there exists* $0<h\leqslant a$ *such that on the interval* J *the initial value problem*

$$(10) \qquad \begin{aligned} &y'(x) = F_n(x, y(x), y[f(x)]), \qquad x \in J, \\ &y(x) = \varphi_1(x), \qquad x \in \langle x_0+h, \beta\rangle, \end{aligned}$$

where $\beta = \max_{x \in J} f(x)$ *and* $\varphi_1: \langle x_0+h, \beta\rangle \to \langle y_0-b, y_0+b\rangle$ *is given continuous function, has for every* $n=0, 1, \ldots$, *exactly one solution* y_n. *Moreover, the sequence* $\{y_n\}_{n=1}^{\infty}$ *convergences uniformly on* J *to* y_0.

Proof. Let G_1 be a space of these functions $\varphi : J \to \langle y_0 - b, y_0 + b \rangle$ which fulfil a Lipschitz condition with constant less than or equal to M and fulfilling condition $\varphi(x_0 + h) = \varphi_1(x_0 + h)$. Let X_1 be a space of all functions ψ such that

$$\psi(x) = \begin{cases} \varphi(x), & x \in J, \\ \varphi_1(x), & x \in \langle x_0 + h, \beta \rangle, \end{cases}$$

where $\varphi \in G_1$. We can verify that for $n = 0, 1, \ldots, \psi \in X_1$ the transformation

$$T_n(\psi)(x) = \int_{x_0+h}^{x} F_n(s, \psi(s), \psi[f(s)])\, ds, \qquad x \in J,$$

$$T_n(\psi)(x) = \varphi_1(x), \qquad x \in \langle x_0 + h, \beta \rangle$$

is a contraction mapping from X_1 into X_1, As in the proof of Theorem 2, we can show that the sequence $\{T_n\}_{n=1}^{\infty}$ converges pointwise on X_1 to T_0 and from the Theorem 1 we obtain our assertion.

Now we assume the hypothesis.

(v) The function $f : I \to R$ fulfils the condition

$$|f(x) - x_0| \leqslant |x - x_0| \quad \text{for} \quad x \in I$$

Theorem 4. *Let hypotheses* (i), (iii), (v) *be fulfilled. Then there exists* $0 < h \leqslant a$ *such that on* J *the initial value problem*

$$(11) \qquad \begin{aligned} & y'(x) = F_n(x, y(x), y[f(x)]) \qquad x \in J, \\ & y(x_0) = y_0 \end{aligned}$$

has for every $n = 0, 1, \ldots,$ *exactly one solution* y_n *and the sequence* $\{y_n\}_{n=1}$ *converges uniformly on* J *to* y_0.

Proof. We define X_2 as the space of these functions $\varphi : J \to \langle y_0 - b, y_0 + b \rangle$ which fulfil a Lipschitz condition with constant less than or equal to M. From (v) follows that if $\varphi \in X_2$ than $\varphi[f(x)] \in \langle y_0 - b, y_0 + b \rangle$ for $x \in J$. For $\varphi \in X_2$ and $n = 0, 1, \ldots$ we define the transformation T_n by

$$T_n(\varphi)(x) = y_0 + \int_{x_0}^{x} F_n(s, \varphi(s), \varphi[f(s)])\, ds, \qquad x \in J.$$

We can verify that for each $n = 0, 1, \ldots, T_n$ is a contraction mapping from X_2 into X_2 and that the sequence $\{T_n\}_{n=1}^{\infty}$ converges pointwise on X_2 to T_0. Applying Theorem 1 we complete the proof.

We shall assume

(vi) The sequence $\{F_n\}_{n=1}^{\infty}$ tends uniformly on Y to F_0.

(vii) The functions $f_n : I \to R$, $n = 0, 1, \ldots,$ fulfil the condition

$$|f_n(x) - x_0| \leqslant |x - x_0|, \qquad x \in I.$$

Theorem 5. *Let hypotheses* (i) (vi), (vii) *be fulfilled. Then there exists* $0<h\leqslant a$ *such that on* J *the initial value problem*

$$(12)\qquad \begin{aligned} &y'(x)=F_n(x, y(x), y[f_n(x)]), \qquad x\in J\\ &y(x_0)=y_0 \end{aligned}$$

has for every $n=0, 1, \ldots,$ *exactly one solution* y_n *and the sequence* $\{y_n\}_{n=1}^{\infty}$ *converges uniformly on* J *to* y_0.

The proof of the above theorem does not differ from that given for Theorem 2 and is therefore omitted.

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