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# GRAPH EQUATIONS, GRAPH INEQUALITIES AND A FIXED POINT THEOREM

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## 0. Abstract.

In this paper we propose some graph theory problems to be considered in terms of "graph equations" and "graph inequalities". These notions have already been used in some papers written by the first and the third author and are now formally defined. Applying a fixed point theorem to graph equations and graph inequalities we want to point out that such means as fixed point theorem, typical for analysis, work also in the field of discrete mathematics, although not with so great power, which can be seen from the obtained results (Theorems 1 and 2).

## 1. Some basic definitions.

In this paper only finite undirected graphs without loops or multiple edges are being considered, including also the empty graph $\theta$, i. e. the graph without vertices (or edges). The set of all such graphs will be denoted by $\mathcal{G}$.

The complement $\bar{G}$ of a graph $G$ is a graph having the same vertex set as $G$ and in which two vertices are adjacent if and only if they are not adjacent in $G$. Note, that the complement of the empty graph is the empty graph.

The line graph $L(G)$ of a graph is the graph having the edge set of $G$ as its vertex set and in which two vertices are adjacent if and only if the corresponding edges are adjacent in $G$. Note, that the line graph of the totally disconnected graph or of the empty graph is the empty graph. In other words, the set $\mathcal{G}$ is closed under the operation of taking line graphs while $\mathcal{G}\setminus\{\theta\}$ is not[1)].

As usual we have $L^0(G)=G$, while $L^n(G)=L\left(L^{n-1}(G)\right)$ for any positive integer $n$. Further, $L^{-1}(G)=\{H \mid H\in\mathcal{G},\ L(H)=G\}$, while the meaning of $L^{-n}(G)$ is now evident.

The union $G=\bigcup_{i=1}^{k} G_i$ is a graph whose vertex set is the union of the vertex sets of graphs $G_i$, while two vertices in $G$ are adjacent if and only if the corresponding vertices are adjacent in at least one of the graphs $G_i$.

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[1)] The advantages of introducing the notion of the empty graph in general, are discussed in [1].

By $kG$ we shall denote the union of $k$ copies of the graph $G$. In particular, we have $0 \cdot G = 0$ and $0 \cup G = G$. So, we can write $L\left(\bigcup_{i=1}^{k} G_i\right) = \bigcup_{i=1}^{k} L(G_i)$.

For all other definitions and notations on graphs the reader is referred to [2].

## 2. The notion of graph equation and inequality.

On the set of all graphs various operations are defined. Using those operations one can form, starting from a finite sequence of graphs $G_1, \ldots, G_n$, a compound graph denoted by $f(G_1, \ldots, G_n)$. The expression $f(G_1, \ldots, G_n)$ could be called the "algebraic expression" in variables $G_1, \ldots, G_n$. Now, if we assume that two graphs are equal, if and only if they are isomorphic (isomorphism is, naturally, an equivalence relation), by equating two algebraic expressions having the same set of variables, we shall get a relation which will be called a graph equation. In general case it can be written in the form:

$$(1) \qquad f(G_1, \ldots, G_n) = g(G_1, \ldots, G_n)^{2)}.$$

Then, one can pose the problem of solving such an equation, i. e. of finding all $n$-tuples $(G_1, \ldots, G_n)$ of graphs satisfying (1).

If $\leq$ denotes a partial order relation in $\mathcal{G}$, then

$$(2) \qquad f(G_1, \ldots, G_n) \leq g(G_1, \ldots, G_n)$$

is called a graph inequality.

For example, the relation "to be an induced subgraph of" is a partial order relation and it will be denoted by $\subseteq$.

Finally let us note that the systems of graph equations or inequalities can be considered, too.

## 3. Some examples of graph equations.

We shall now mention some known results reformulating them according to our terminology.

The solutions of the equation $\overline{G} = G$ are self-complementary graphs. It is known that there are infinitely many such graphs.

The existence and uniqueness of the solution of equation $L(G) = H$, where $H$ is a given graph was investigated by L. Beineke and H. Whitney. In [3], L. Beineke proved that the equation $L(G) = H$ has solutions, if and only if $H$ does not contain any one of nine graphs from fig. 1 as an induced subgraph. For the same equation H. Whitney proved in [4] that if both graphs $G$ and $H$ $(\neq 0)$ are connected, then for each $H$, $G$ is unique (if it exists) except for $H = K_3$, when $G$ could be $K_3$ or $K_{1,3}$.

[2] Naturally, a graph equation can be defined more generally, but now we shall not deal with that.

Also, we shall only mention that the equation $L(G)=H$ has been treated in many special cases: for instance, if $H$ belongs to the set of regular graphs, bipartite graphs and so on.

For the equation $T(G)=H$, where $T(G)$ is the total graph of a graph $G$, the analogous results to the above ones were stated by M. Behzad and H. Radjavi in [5] and [6].

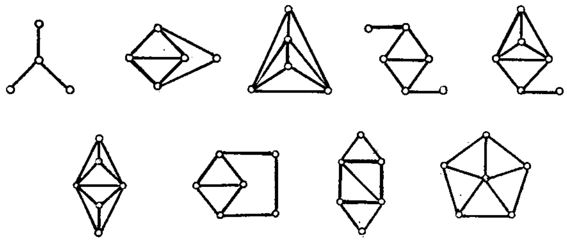

Fig. 1

In [7] D. Cvetković and S. Simić have described all graphs $H$ without triangles satisfying the equation $\overline{L(G)}=H$. Notice that the list of forbidden induced subgraphs for the graph $H$ coincides with the list of complements of the graphs from fig. 1.

In [8] and [9] V. V. Menon pointed out that, for any $n$, $G(\neq\theta)$ is the solution of the equation $L^n(G)=G$, if and only if $G$ is a regular graph of degree two. Of course, the empty graph is also a solution.

Also the equation $L^n(G)=\overline{G}$ has $G=\theta$ as a solution for any $n$. For $n=1$ M. Aigner [10] pointed out that the only other solutions are graphs from fig. 2.

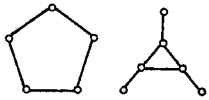

Fig. 2

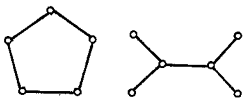

Fig. 3

In [11] S. Simić proved that for $n=2$ the only solutions different from $\theta$ are graphs from fig. 3, while for $n\geq 3$ only the cycle of length five and $\theta$ are solutions.

In [12] D. Cvetković and S. Simić solved the equations $L(G)=T(H)$ and $\overline{L(G)}=T(H)$.

Finally, let us mention that in [13] L. Beineke, in fact, has solved the system of graph equations $L(G_1)=H$, $\overline{L(G_2)}=H$. A graph $H$ satisfying the above system is called coderived graph.

## 4. A fixed point theorem.

Let $A$ (see [14] or [15]) throughout this section be a set having the following properties:

1) $A$ is partially ordered by the relation $\leq$ and has an element $\varnothing$ with the property that $\varnothing\leq u$ holds for every $u\in A$.

2) If $(u_i)_{i \in N}$ is a sequence of elements from $A$ such that $u_{i+1} \leq u_i$ holds for every $i \in N$, then, it is possible to correspond to that sequence a unique element $u \in A$ (which is denoted by $u_i \searrow u$ and $u$ is called the limiting value of $(u_i)$) with the following properties:

a) if $u_i = u$ for every $i \in N$, then $u_i \searrow u$,

b) if $u_i \searrow u$, $v_i \searrow v$ and $u_i \leq v_i$ for all $i \in N$, then $u \leq v$,

c) $u$ remains the same if a finite number of members of $(u_i)_{i \in N}$ are replaced by arbitrary elements from $A$.

For a mapping $\varphi : A \to A$ we shall say that it is nondecreasing, if $u \leq v$ $(u, v \in A)$ implies $\varphi(u) \leq \varphi(v)$. A mapping $\varphi$ is continuous with respect to monotonic sequences if $u_i \searrow u$ implies $\varphi(u_i) \searrow \varphi(u)$.

Let $u_0, v_0 \in A$ and $u_0 \leq v_0$. The segment $[u_0, v_0]$ in $A$ is the set of all elements $w \in A$, such that $u_0 \leq w \leq v_0$. Also, we shall consider the equation

$$\varphi(x) = x, \tag{3}$$

where $x \in A$ and $\varphi$ is the mapping on $A$ already defined. The solution $x^* \in [u_0, v_0]$, of equation (3) is maximal on segment $[u_0, v_0]$, if for every solution $y \in [u_0, v_0]$ of the same equation we have $y \leq x^*$.

Now, according to [14] or [15], the following theorem holds.

Theorem A. *Let $\varphi : A \to A$ be a mapping defined on segment $\Delta = [\varnothing, \varkappa] \subset A$. Also, let $\varphi$ be nondecreasing, continuous with respect to monotonic sequences, $\varphi(u) \in A$ and for some $b \in \Delta$ let $\varphi(b) \leq b$. Then the equation $\varphi(x) = x$ has on the segment $[\varnothing, b]$ the maximal solution $m(\varphi, b)$. If $\varnothing \leq a \leq b$ and $a \leq \varphi(a)$, then $a \leq m(\varphi, b)$.*

The proof of the above theorem gives a possibility of the approximate computing the solutions of equation (3) if we know the solutions of the corresponding inequality. Namely, the sequences of the type $(\varphi^i(b))_{i \in N}$ converge to the solution $m(\varphi, b)$.

In some special cases the above theorem can be used in the opposite sense, namely for finding solutions of inequalities, if the solutions of corresponding equations are known.

First, we shall introduce some notations. For a mapping $\varphi : A \to A$ and $B \subset A$ let $\varphi^{-1}(B)$ be the inverse image of $B$, i. e.

$$\varphi^{-1}(B) = \{x \mid x \in A, \varphi(x) \in B\}.$$

Then let

$$\bar{\varphi}(B) = B \cup \varphi^{-1}(B) \cup \varphi^{-2}(B) \cup \cdots = \bigcup_{i=0}^{+\infty} \varphi^{-i}(B).$$

Now, the following theorem can be easily proved.

Theorem B. *Let $A$ be a set, which besides the foregoing conditions, satisfies the conditions that each (nonincreasing) sequence of the type $a$, $\varphi(a)$, $\varphi^2(a), \ldots, (a \in A)$ becomes constant starting from one member, where the mapping $\varphi : A \to A$ is nondecreasing and continuous with respect to monotonic sequences. Also, let $S$ and $T$ be the sets of solutions of equation $\varphi(x) = x$ and inequality $\varphi(x) \leq x$, respectively. Then $T \subset \bar{\varphi}(S)$ holds.*

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[3] In [14] or [15] theorem A is considered as a lemma on functional inequalities.

Proof. Let $y \in T$, which means that $\varphi(y) \leq y$. Then from the suppositions we have that the sequence $y, \varphi(y), \varphi^2(y), \ldots$ is nonincreasing and it becomes constant starting from one member (i. e. for some $n$ we have that $\varphi^{i+n}(y) = \varphi^n(y)$ $(i = 0, 1, 2, \ldots)$). Therefrom we have $\varphi^{n+1}(y) = \varphi^n(y)$ or in other words, $\varphi(z) = z$, where $z = \varphi^n(y)$. Hence, $z \in S$ and $y \in \varphi^{-n}(z)$. So we have $y \in \bar{\varphi}(S)$.

## 5. Graph equations and a fixed point theorem.

Obviously, $(\mathcal{G}, \subseteq)$ is a partially ordered set with the least element $\theta$. Owing to the discrete structure of the set $\mathcal{G}$ each nonincreasing sequence becomes constant starting from one member. Hence, there is no special need for introducing the notion of limiting value for a sequence of graphs, since it is determined in natural manner and is equal to the constant member which appears in each nonincreasing sequence of graphs.

The following relations can be easily verified

$$G_1 \subseteq G_2 \Rightarrow \bar{G}_1 \subseteq \bar{G}_2, \quad G_1 \subseteq G_2 \Rightarrow L(G_1) \subseteq L(G_2).$$

Therefrom we get the relations

$$G_1 \subseteq G_2 \Rightarrow L^n(G_1) \subseteq L^n(G_2), \quad G_1 \subseteq G_2 \Rightarrow \overline{L^n(G_1)} \subseteq \overline{L^n(G_2)},$$

where $n$ is a nonnegative integer.

Now, it can be easily seen that Theorem A may be (at least in principle) used for determining the solutions of graph equations

$$L^n(G) = G, \tag{4}$$

$$L^n(G) = \bar{G}, \tag{5}$$

where $n$ is a positive integer. But the application of Theorem A in this direction is met with difficulties arising from the complexity of graphs structure. Namely, in order to find all solutions of the above equations by the means of Theorem A it is necessary to find (if any additional observations are not made) all solutions of graph inequalities $L^n(G) \subseteq G$ and $L^n(G) \subseteq \bar{G}$. Unfortunately, it is known in graph theory that the problem of establishing whether a graph is an induced subgraph of another one is more difficult than the so called isomorphism problem. Hence, on the basis of the present situation the application of a fixed point theorem in solving graph equations seems to be nonefficient. However, the situation is better with graph inequalities as we shall show in the next section.

## 6. Solutions of some graph inequalities.

Now we shall solve, by means of the introduced apparatus, the following graph inequalities:

$$L^n(G) \subseteq G, \tag{6}$$

$$L^n(G) \subseteq \bar{G}, \tag{7}$$

where $n$ is a positive integer.

Primarily, let us notice some interesting properties of the operation $L$. In the set $\mathcal{G}$ the operation $L$ and its inverse $L^{-1}$ act similarly as operators of differentiation $D$ and integration $D^{-1}$ in some set of functions. A graph $lK_1 = lP_1$ $(l \geq 0)$ plays the role[4] of an integration constant. Quite analogously, $L^n$ and $L^{-n}$ can be brought in the correspondence with $D^n$ and $D^{-n}$. In the case of $L^{-n}$, graph $P^n = \bigcup_{i=1}^{n} l_i P_i \, (l_i \geq 0)$ plays the role of a polynomial of the $(n-1)$-th degree with undetermined coefficients. In the further text the set of all such graphs $P^n$ will be denoted by $\mathcal{P}_n$. For example[5], $L^{-3}(C_5) = C_5 \cup P^3$, where $P^3$ is an arbitrary element of

$$\mathcal{P}_3 = \{l_1 K_1 \cup l_2 K_2 \cup l_3 P_3 \mid l_1, l_2, l_3 \geq 0\}.$$

Let us first consider inequality (6). Now, we have $\varphi = L^n$. The general solution $G^*$ of equation (4) according to the foregoing can be written in the form $G^* = \bigcup_{i=3}^{m} l_i C_i$ where $m \geq 3$ and $l_i \geq 0$. So, we have set $S$ from Theorem B. Now, it is not difficult to find the family of graphs $L^{-kn}(G^*)$ $(k = 1, 2, \ldots)$. Having in view the adopted notation it follows that $\varphi^{-k}(G^*) = L^{-kn}(G^*) = G^* \cup P^{kn}$, where $P^{kn} \in \mathcal{P}_{kn}$ $(k = 0, 1, 2, \ldots)$. Now, we must check which elements of $\bar{\varphi}(G^*)$ really satisfy (6). It can easily be seen that

$$L^n(G^* \cup P^{kn}) \subseteq G^* \cup P^{kn}$$

holds for each positive integer $n$ and each nonnegative integer $k$.

So, we have proved the following theorem.

Theorem 1. *Graph $G$ is a solution of graph inequality $L^n(G) \subseteq G$ if and only if its greatest vertex degree does nos exceed 2, i. e. if and only if*

$$G = \left( \bigcup_{i=3}^{m} l_i C_i \right) \cup P^l,$$

*where $m \geq 3$ and $P^l \in \mathcal{P}_l$ for some $l \geq 1$.*

Remark. If $G_1 \to G_2$ denotes that $G_1$ is a subgraph (not necessarily induced) of $G_2$, then $G_1 \to G_2$ again implies $L^n(G_1) \to L^n(G_2)$ for each $n$. Applying the same arguments it follows that inequality $L^n(G) \to G$ has the same solutions as (6).

Let us consider now inequality (7). If we start from the solutions of equation (5) we easily find the iterated inverse images of the mapping $\varphi = \overline{L^n}$. In order to decrease the number of involved operations it is useful to have in mind the mentioned results about equation $\overline{L(G)} = H$ from [7].

[4] By $K_n$, $C_n$, $P_n$ we understand the complete graph, the cycle and the path with $n$ vertices, respectively. $K_{m,n}$ denotes the bicomplete graph with $m+n$ vertices.

[5] Like as in the integral calculus we shall write $L^{-3}(C_5) = C_5 \cup P^3$ instead of $L^{-3}(C_5) = \{C_5 \cup P^3 \mid P^3 \in \mathcal{P}_3\}$.

Especially, the complement of a line graph must not have, as induced subgraphs, the following graphs: $C_3 \cup K_1$, $K_2 \cup K_{1,2}$, $K_2 \cup 3K_1$, $C_4 \cup 2K_2$, $C_5 \cup K_1$ (these are complements of some graphs from fig. 1). Fortunately the process of finding the iterated inverse images for the operation $\overline{L^n}$ finishes very quickly in majority of cases. So we have arrived at the following theorem whose proof will be omitted although it is not quite short.

Theorem 2. *Graph inequality* $L^n(G) \subseteq \overline{G}$ *has the following solutions:*

1) *graphs from fig.* 4 *if* $n=1$;

2) $P_{n+1}$, $P_{n+2}$, $P_{n+3}$, $P_{n+4}$, $2P_{n+1}$, $P_{n+1} \cup P_{n+2}$, $K_{1,3}$, $K_{1,3} \cup P_{n+1}$, $C_3 \cup 2K_2$, $C_5$, $\theta$ if $n \geq 1$;

3) *graphs from fig.* 5 *if* $n=2$;

4) $C_3 \cup K_1 \cup K_2$, $C_3 \cup 2K_2$, $C_3 \cup P_i$ $(i=3, \ldots, n)$, $C_4 \cup P_i$ $(i=2, \ldots, n)$ *if* $n \geq 2$;

5) *graphs from fig.* 6 *if* $n=3$.

*If* $G_0$ *is a solution then* $G = G_0 \cup P^n$, *where* $P^n \in \mathcal{P}_n$, *is again a solution. There exist no other solutions.*

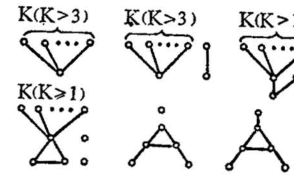

Fig. 4

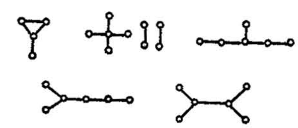
Fig. 5


Fig. 6

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