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ON THE DECOMPOSITION OF ARBITRARY SECOND ORDERED STOCHASTIC PROCESS

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We shall assume that the stochastic process $X=\{X(t),\ t\in R\}$ and related concepts are defined as in [2]; all notions concerning measure theory or theory of linear operators in Hilbert space are defined as in [5]. We shall not assume that any special condition for the stochastic process X is satisfied.

Theorem 1. *There is the unique decomposition of the stochastic process X to the orthogonal sum of the stochastic processes $X_1=\{X_1(t),\ t\in R\}$ and $X_2=\{X_2(t),\ t\in R\}$, such that the following two conditions are satisfied:*

(a) $\mathcal{H}(X_1; t-0)=\mathcal{H}(X_1: t)$ for each $t\in R$;

(b) X_2 is the stochastic process with discrete inovations (i.e. all its orthogonal inovations are discrete).

Proof. Let

$$x_t=X(t)-P_{\mathcal{H}(X;\, t-0)}X(t),\quad t\in R, \tag{1}$$

(for any subspace $\mathcal{H}_1\subset\mathcal{H}(X)$, $P_{\mathcal{H}_1}$ denotes the projection operator from $\mathcal{H}(X)$ onto $\mathcal{H}_1$) and let $\mathcal{S}_2$ be the smallest subspace of $\mathcal{H}(X)$ generated by $x_t,\ t\in R$:

$$\mathcal{S}_2=\overline{\mathcal{L}}\{x_t,\ t\in R\};$$

let $\mathcal{S}_1$ be the subspace of $\mathcal{H}(X)$ such that the equality

$$\mathcal{S}_1\oplus\mathcal{S}_2=\mathcal{H}(X)$$

holds. Let us define the stochastic processes X_1 and X_2 by

$$X_1(t)=P_{\mathcal{S}_1}X(t),\quad X_2(t)=P_{\mathcal{S}_2}X(t),\quad t\in R.$$

It is easy to see that we have

$$X(t)=X_1(t)+X_2(t),\quad t\in R, \tag{2}$$

and

$$\big(X_1(u),\quad X_2(v)\big)=0 \text{ for all } u,\ v\in R.$$

From the equality $\mathcal{H}(X_1; t)=P_{\mathcal{S}_1}\mathcal{H}(X; t),\ t\in R$, it follows

$$\mathcal{H}(X_1; t)=P_{\mathcal{S}_1}\mathcal{H}(X; t-0)\oplus P_{\mathcal{S}_1}\overline{\mathcal{L}}\{x_t\},\quad t\in R,$$

where x_t is defined by (1). However, $x_t \in \mathcal{S}_2$ and this means that $\overline{\mathcal{L}}\{x_t\}$ is orthogonal to $\mathcal{S}_1$, i.e. $P_{\mathcal{S}_1}\overline{\mathcal{L}}\{x_t\}=0$; hence

$$\mathcal{H}(X_1; t)=P_{\mathcal{S}_1}\mathcal{H}(X; t-0)=\mathcal{H}(X_1; t-0), \quad t \in R,$$

and (a) is satisfied.

The equality

$$X_2(t)=P_{\overline{\mathcal{L}}\{x_s,\ s \leq t\}} X(t), \quad t \in R. \tag{3}$$

holds because $X(t)$ is orthogonal to x_u for all $u>t$. From this fact it follows that all inovations of stochastic process X_2 are of the form $X_2(t)-P_{\mathcal{H}(X_2; t-0)}X_2(t)$, and each inovation of this form is, because of (3), equal to $P_{\overline{\mathcal{L}}\{x_t\}}X_2(t)$. That means that (b) is satisfied.

It is easy to show the uniquerness of the decomposition (2). Let us suppose that there exists another decomposition,

$$X(t)=X_1^*(t)+X_2^*(t), \quad t \in R,$$

of the process X, such that the conditions (a) and (b) are satisfied for the processes X_1^* and X_2^*, respectively. In this case, according to (2) and (4), we have

$$X_1(t)-X_1^*(t)=X_2(t)-X_2^*(t), \quad t \in R.$$

But, the last equality is false, because $X_2-X_2^*=\{X_2(t)-X_2^*(t),\ t \in R\}$ is the process with discrete inovations and the condition (a) is satisfied for the process $X_1-X_1^*=\{X_1(t)-X_1^*(t),\ t \in R\}$. QED

Theorem 2. *Let X be an arbitrary stochastic process satisfying the equality*

$$\mathcal{H}(X; t-0)=\mathcal{H}(X; t) \text{ for all } t \in R. \tag{4}$$

That process can be represented as orthogonal sum of the stochastic process with homogenous maximal spectral types[1]*; these spectral types are mutually orthogonal and they have mutually different multiplicities. This decomposition of X is unique.*

This theorem is the immediate consequence of the theorems 10.4.11 and 10.4.15 of [5].

Theorem 3. *Let X be an arbitrary stochastic process satisfying* (4). *It can be decomposed to the orthogonal sum*

$$X(t)=X_1(t)+X_2(t), \quad t \in R,$$

where the following conditions 1°, 2°, *and* 3° *are satisfied:*

1° a) *Maximal spectral type of the process* X_1 *is ordinary*[2].

b) *The inequality* mult $\rho \leqslant \aleph_0$ *holds for each spectral type* ρ *belonging to the process* X_1[2].

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[1] Maximal spectral type of the stochastic process is, by definition, equal to the maximal spectral type of the resolution of the identity of this stochastic process; about "maximal spectral type of the resolution of the identity" (which can be ordinary or generalized) see, for example, [5] or [4].

[2] We say that the spectral type ρ belongs to the stochastic process X if it belongs to the resolution of the identity of X; see, for example, [5] or [4].

2° *At least one of the following two conditions is satisfied:*

a) *Maximal spectral type of the process* X_2 *is generalized and for any* $z \in \mathcal{H}(X_2)$, *if mult* $\rho_z \leq \aleph_0$, *then there exist continuously many elements* $z_\lambda \in \mathcal{H}(X_2)$, $\lambda \in \Lambda$, *such that* $\rho_{z_\lambda} \perp \rho_{z_\nu}$ $(\lambda \neq \nu)$ *and* $\rho_{z_\lambda} \perp \rho_z$ *for all* $\lambda, \nu \in \Lambda$.

b) *The equality mult* $\sigma = \aleph_1$ *holds for each spectral type* σ *belonging to the process* X_2.

3° *If the arbitrary spectral types* ρ *and* σ *belong to the processes* X_1 *and* X_2, *respectively, then* ρ *is orthogonal to* σ *and mult* $\rho \neq$ *mult* σ.

The described decomposition of the process X is unique.

Proof. The preceding theorem has the basic role in the proof. According to this theorem, the space $\mathcal{H}(X)$ can be decomposed to the ortogonal sum of the subspaces $\mathcal{H}_\lambda$, whose maximal spectral types[3)] are mutually orthogonal and multiplicities m_λ of these spectral types are mutually different. Let $\mathcal{H}_1$ be the orthogonal sum of those subspaces $\mathcal{H}_\lambda$ whose maximal spectral types are ordinary and have multiplicities not greater than $\aleph_0$; let $\mathcal{H}_2 = \mathcal{H}(X) \ominus \mathcal{H}_1$. The subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ reduce the resolution of the identity $E_X = \{E_X(t),\ t \in R\}$ of the process X. The equality

$$X(t) = P_{\mathcal{H}_1} X(t) + P_{\mathcal{H}_2} X(t)$$

holds for all $t \in R$. Let us put: $X_1(t) = P_{\mathcal{H}_1} X(t)$, $X_2(t) = P_{\mathcal{H}_2} X(t)$, $t \in R$. The stochastic processes X_1 and X_2 are orthogonal. From the equality $\mathcal{H}_1 = \mathcal{L}\{X_1(t),\ t \in R\}$ it follows that the condition 1° is satisfied. If the maximal spectral type of X is generalized, then, according to 1° a), the maximal spectral type of X_2 will be generalized and the condition 2° a) will be satisfied. If the maximal spectral type of X is ordinary and dim $\mathcal{H}(X) = \aleph_1$, then the maximal spectral type of X_2 will be ordinary, but it will be homogeneous and its multiplicity will be equal to $\aleph_1$; hence, in this case the condition 2° b) will be satisfied. The uniqueness of the decomposition follows from the described construction. QED

From the theorems 1 and 3 it follows immediately the following.

Theorem 4. *Let X be an arbitrary second ordered stochastic process. There exists the unique decomposition of this process to the orthogonal sum of the stochastic processes* X_1, X_2 *and* X_3 *such that:*

1° $\mathcal{H}(X_1; t-0) = \mathcal{H}(X_1; t)$ *for each* $t \in R$ *and* $\dim \mathcal{H}(X_1) \leq \aleph_0$;

2° $\mathcal{H}(X_2; t-0) = \mathcal{H}(X_2; t)$ *for each* $t \in R$;

3° X_3 *is the stochastic process with discrete inovations.*

It is easy to see that the processes X_1 and X_2 can be studied by the methods of linear operators in Hilbert spaces, because they determine the resolution of the identity in the corresponding Hilbert spaces. The process X_3 cannot be studied by the same methods; the "white noise" is one of processes of this kind.

Now, let X_1 and X_2 be the stochastic processes which satisfy the conditions 1° and 2° of the preceding theorem, respectively. The difference between

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3) Maximal spectral type of the arbitrary subspaces $\mathcal{H}_\lambda$ is, by definition, equal to the maximal spectral type of the stochastic process which is obtained as the projection of the stochastic process X onto $\mathcal{H}_\lambda$.

these two processes is in the following. The space $\mathcal{H}(X_1)$ is obviously separable and, because of that, it can be studied as in [1—3] (it is not necessary to make the assumption about the existence of left and right limits of process for values of t). In the following example we will see that the space $\mathcal{H}(X_2)$ can be non-separable (i.e. it can be dim $\mathcal{H}(X_2)=\aleph_1$).

Example. By $Z_\lambda=\{Z_\lambda(t),\ t\in R\}$, $\lambda\in\Lambda$ (card $\Lambda=\aleph_1$) we shall denote continuous (in the sense of [1]) stochastic process with orthogonal increments. We shall suppose that the processes Z_λ, $\lambda\in\Lambda$, are mutually orthogonal and that the (maximal) spectral type of each of them is equal to ρ (of course, it is well known that the multiplicity of a continuous stochastic process with orthogonal increments is equal to 1). It is known ([1]) that, for each $\lambda\in\Lambda$, there exists a countable set $\{t_1, t_2, \ldots\}$ of values of t, such that the linear manifold spanned by $Z_\lambda(t_1)$, $Z_\lambda(t_2), \ldots$ is everywhere dense in $\mathcal{H}(Z_\lambda)$, i.e. $\mathcal{H}(Z_\lambda)=\overline{\mathcal{L}}\{Z_\lambda(t_i),\ i=1, 2, \ldots\}$. The last equality holds for each countable everywhere dense set in R.

Let q be the fixed irrational number and A — the set of all numbers of the form $m+nq$, where m and n are arbitrary integers: $A=\{m+nq\mid m, n$ — integers$\}$. Let us define the relation $\sim$ on R in the following way: $t_1\sim t_2$ if and only if $t_1-t_2\in A$. It is easy to see ([6]) that $\sim$ is an equivalent relation on R, that there is continuously many equivalent classes R_λ in R, and that each equivalent class is countable and everywhere dense in R.

Let us denote by φ the following one-to-one mapping from Λ onto the set of all equivalent classes: $\varphi(\lambda)=R_\lambda$, $\lambda\in\Lambda$. The linear manifold spanned by the set $\{Z_\lambda(t_i),\ t_i\in R_\lambda\}$ is everywhere dense in $\mathcal{H}(Z_\lambda)$.

We shall define the stochastic process X_2 by the equality

$$X_2(t)=Z_\lambda(t),\quad t\in R_\lambda,\quad \lambda\in\Lambda.$$

It is clear that the process X_2 is well defined, because the classes R_λ are disjoint and $\bigcup_{\lambda\in\Lambda} R_\lambda=R$. Also, it is easy to show that the equality

$$\mathcal{H}(X;t)=\sum_{\lambda\in\Lambda}\oplus\,\mathcal{H}(Z_\lambda;t) \tag{5}$$

holds for all $t\in R$ (the addition is formally, because card $\Lambda=\aleph_1$). From (5) and from the continuity of the process Z_λ for each $\lambda\in\Lambda$, it follows that X_2 satisfies the condition 2° of the theorem 4, but dim $\mathcal{H}(X_2)=\aleph_1$.

The study of the stochastic process X, which satisfies only the condition (4), will be the subject under discussion of one later paper.

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