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# A NOTE ON M. PREŠIĆ'S METHOD

*Miroslav D. Ašić, Vera V. Kovačević*

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In [2] an iterative method for finding $k$ roots of a polynomial $P(x)$ of degree $n$, where $1 \leqslant k \leqslant n$, is proposed. The formulas at the $i$-th step are:

$$a_j(i+1) = a_j(i) - \frac{P(a_j(i))}{\prod_{\substack{s=1 \\ s \neq j}}^{k} (a_j(i) - a_s(i))\, Q_i(a_j(i))}, \quad j = 1, \ldots, k \tag{1}$$

where

$$P(x) = (x - a_1(i)) \cdots (x - a_k(i))\, Q_i(x) + R_i(x). \tag{2}$$

The following result is proved:
If $\xi_1, \ldots, \xi_k$ are roots of $P(x)$ and $P'(\xi_1) \cdots P'(\xi_k) \neq 0$, then there exists a neighbourhood $U$ of $\mathbf{a} = (\xi_1, \ldots, \xi_k)^T$ such that the sequence $(\mathbf{a}(i)) = ((a_1(i), \ldots, a_k(i))^T)$ generated by (1) converges to $\mathbf{a}$ if $(a_1(0), \ldots, a_k(0))^T \in U$. Furthermore, the convergence is quadratic.

The purpose of this note is to show that this method belongs to the class of quasi-Newton methods.

Definition 1. A method for finding a zero of a twice continuously differentiable function $\boldsymbol{F}: R^k \to R^k$ is called a quasi-Newton method if the following properties hold (see [1]):

(i) $\mathbf{a}(i+1) = \mathbf{a}(i) - \sigma(i)\, \mathbf{s}(i)$, $i = 0, 1, \ldots$; $\sigma(i) > 0$ and $\mathbf{s}(i) \in R^k$

(ii) If the sequence $(\mathbf{a}(i))$ converges to a point $\mathbf{a}$ such that $\mathbf{F}(\mathbf{a}) = \mathbf{0}$ and $\det \mathbf{J}(\mathbf{a}) \neq 0$ then

$$\frac{1}{\|\mathbf{g}(i)\|} \|\sigma(i)\, \mathbf{s}(i) - \mathbf{J}^{-1}(\mathbf{a}(i))\, \mathbf{g}(i)\| = O(\|\mathbf{g}(i-\nu)\|),$$

where $0 \leqslant \nu \leqslant k$, $\mathbf{J}(\mathbf{x})$ is the Jacobian of $\mathbf{F}$ and $\mathbf{g}(i) = \mathbf{F}(\mathbf{a}(i))$.

In order to obtain M. Prešić's method we formulate the following problem: find a zero of $\mathbf{F}(\mathbf{x})=(P(x_1),\ldots,P(x_k))^T$ and in order to solve it, use the formulas

$$\mathbf{a}(i+1)=\mathbf{a}(i)-\sigma(i)\,\mathbf{s}(i),\quad i=0,1,\ldots$$

$$(1')\qquad \sigma(i)=1,\ i=0,1,\ldots$$

$$\mathbf{s}(i)=\mathbf{A}(\mathbf{a}(i))\,\mathbf{J}^{-1}(\mathbf{a}(i))\,\mathbf{g}(i),\ i=0,1,\ldots,$$

where

$$\mathbf{A}(\mathbf{a}(i))=diag\left\{\frac{P'(a_1(i))}{P'(a_1(i))-R_i'(a_1(i))},\ldots,\frac{P'(a_k(i))}{P'(a_k(i))-R_i'(a_k(i))}\right\}$$

($P(x)$ and $R_i(x)$ being the same as in (2)).

Note that $\mathbf{A}(\mathbf{a}(i))$ is well defined in some neighbourhood of $\mathbf{a}$ since $P'(a_j(i))$ is close to $P'(\xi_j)\neq 0$ and $R_i'(a_j(i))$ is close to zero if $\|\mathbf{a}(i)-\mathbf{a}\|$ is small enough. It is clear that the method (1′) satisfies condition (*i*) of Definition 1. In order to show that it satisfies the condition (*ii*) suppose that

$$\mathbf{a}(i)\to\mathbf{a},\ i\to\infty,\ \mathbf{F}(\mathbf{a})=\mathbf{0}\ \text{and}\ \det\mathbf{J}(\mathbf{a})\neq 0.$$

We have

$$\frac{1}{\|\mathbf{g}(i)\|}\|\mathbf{A}(\mathbf{a}(i))\mathbf{J}^{-1}(\mathbf{a}(i))\mathbf{g}(i)-\mathbf{J}^{-1}(\mathbf{a}(i))\mathbf{g}(i)\|\leqslant\|\mathbf{J}^{-1}(\mathbf{a}(i))\|\cdot\|\mathbf{A}(\mathbf{a}(i))-\mathbf{I}\|=$$

$$(3)\qquad =\|\mathbf{J}^{-1}(\mathbf{a}(i))\|\cdot\left\|diag\left\{\frac{R_i'(a_1(i))}{P'(a_1(i))-R_i'(a_1(i))},\ldots,\frac{R_i'(a_k(i))}{P'(a_k(i))-R_i'(a_k(i))}\right\}\right\|.$$

The coefficients of $R_i(x)$ can be obtained by setting $x=a_1(i),\ldots,x=a_k(i)$ in (2). The determinant of this system is a Vandermonde determinant and the solutions are linear combinations of $P(a_1(i)),\ldots,P(a_k(i))$ with coefficients which depend on $i$ but are bounded, since $\mathbf{a}(i)\to\mathbf{a},\ i\to\infty$. Hence the coefficients of $R_i'(x)$ are also linear combinations of $P(a_1(i)),\ldots,P(a_k(i))$ with bounded coefficients and as $(\mathbf{a}(i))$ is bounded we conclude that

$$(4)\qquad |R_i'(a_j(i))|\leqslant c\sum_{s=1}^{k}|P(a_s(i))|.$$

Relations (3) and (4) imply that the condition (*ii*) is also satisfied, with $\nu=0$.

Let us show that the sequence generated by (1′) is convergent if $\|\mathbf{a}(0)-\mathbf{a}\|$ is small enough. We have

$$\mathbf{a}(i+1)=\mathbf{a}(i)-\mathbf{A}(\mathbf{a}(i))\mathbf{J}^{-1}(\mathbf{a}(i))\mathbf{g}(i)$$

or

$$\mathbf{a}(i+1)-\mathbf{a}=\mathbf{a}(i)-\mathbf{a}-\mathbf{A}(\mathbf{a}(i))\mathbf{J}^{-1}(\mathbf{a}(i))\,[\mathbf{F}(\mathbf{a}(i))-\mathbf{F}(\mathbf{a})].$$

By Taylor's formula

$$\mathbf{a}(i+1)-\mathbf{a}=\mathbf{a}(i)-\mathbf{a}-\mathbf{A}(\mathbf{a}(i))\mathbf{J}^{-1}(\mathbf{a}(i))[\mathbf{J}(\mathbf{a})(\mathbf{a}(i)-\mathbf{a})+O(\|\mathbf{a}(i)-\mathbf{a}\|^2)\mathbf{e}],$$

where $\mathbf{e}=(1,\ldots,1)^T$, and hence

$$(5)\qquad \|\mathbf{a}(i+1)-\mathbf{a}\|\leqslant\|\mathbf{a}(i)-\mathbf{a}\|\,[\|\mathbf{I}-\mathbf{A}(\mathbf{a}(i))\mathbf{J}^{-1}(\mathbf{a}(i))\mathbf{J}(\mathbf{a})\|+K\|\mathbf{a}(i)-\mathbf{a}\|].$$

As $\mathbf{A}(\mathbf{a})=\mathbf{I}$ and $\mathbf{A}(\mathbf{x})\,\mathbf{J}^{-1}(\mathbf{x})\,\mathbf{J}(\mathbf{a})$ is continuous at $\mathbf{a}$, there exists a ball $B_\varepsilon(\mathbf{a})$ such that

$$\|\mathbf{I}-\mathbf{A}(\mathbf{x})\,\mathbf{J}^{-1}(\mathbf{x})\,\mathbf{J}(\mathbf{a})\|+K\|\mathbf{x}-\mathbf{a}\|\leqslant\frac{1}{2}\ \text{for}\ \mathbf{x}\in B_\varepsilon(\mathbf{a}).$$

If $\mathbf{a}(0)\in B_\varepsilon(\mathbf{a})$ from (5) it follows that $\mathbf{a}(i)\in B_\varepsilon(\mathbf{a})$ for $i=1, 2, \ldots$, and furthermore that $(\mathbf{a}(i))$ converges to $\mathbf{a}$. According to Theorem 2 in [1] the convergence is quadratic.

Global convergence theorem cannot be obtained without further assumptions as can be seen from the following examples:

Example 1 The polynomial $P(x)=x^5-3x^3-2x$ is square-free and has three real roots. Let $\mathbf{a}(0)=(1,-1)^T$. Then $\mathbf{a}(1)=(-1,1)^T$, $\mathbf{a}(2)=(1,-1)^T=\mathbf{a}(0)$ and the sequence $(\mathbf{a}(i))$ diverges.

Example 2 The polynomial $P(x)=8x^2-1$ has two distinct real roots. Let $\mathbf{a}(0)=(1/2,\ 1/4)^T$. Then $\mathbf{a}(1)=(0,\ 0)^T$ and $\mathbf{a}(2)$ is not defined.

## REFERENCES

[1] G. P. Mc Cormick and K. Ritter, *Methods of conjugate directions versus quasi-Newton methods*, Mathematical Programming Vol 3, № 1 (1972).

[2] M. Prešić, *Un procédé itératif pour déterminer k zéros d'un polynome*, C. R. Acad. Sci. Paris T. 273 (1971).