PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 20 (34), 1976, pp. 243–247

REFLEXIVE BANACH SPACE AND FIXED POINT THEOREMS

Milan R. Tasković

(Communicated July 17. 1974)

Introduction

Let X be a reflexive Banach space and let K be a nonempty bounded, closed and convex subset of X. We say that the mapping $f: R^k \rightarrow R$ ($k \in N$, R the set of real numbers) is semihomogeneous iff $f(\lambda x_1, \ldots, \lambda x_k) \leqq \lambda f(x_1, \ldots, x_k)$, $\lambda \geqq 0$.

Definition 1. *A mapping T of K into itself is said to be an f_{RBS-} contraction iff for every $x, y \in K$ there exist nonnegative real numbers $\alpha_i(x, y) = \alpha_i$ $(i = 1, 2, \ldots, 5)$ such that*

$$\| Tx - Ty \| < f(\alpha_1 \| x - y \|, \alpha_2 \| x - Tx \|, \alpha_3 \| y - Ty \|, \alpha_4 \| x - Ty \|, \alpha_5 \| y - Tx \|),$$

$$\sup \{f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5): x, y \in K\} = \lambda \in [0, 1],$$

where the mapping $f: R^5 \rightarrow R$ is increasing, semihomogeneous, and $g(x) = f(\alpha_1 x, \ldots, \alpha_5 x^5)$ is continuous at the point $x = 1$.

In this paper first we establish some fixed point theorems for mappings T of K into itself, which satisfy the property of f_{RBS-} contraction.

Before going to the theorems, we first recollect the following definitions.

Definition 2. ([6], p. 182) *A bounded convex set K in a Banach space X is said to have normal structure if for each convex subset S of K which contains more than one point, there exists $x \in S$ such that $\sup_{y \in S} \| x - y \| < \partial(S)$, $\partial(S)$ being the diameter of S.*

If T is a mapping of K into itself such that for each $x \in K$,

$$\lim \partial [\mathcal{O}(T^n x)] < \partial [\mathcal{O}(x)] \text{ when } \partial [\mathcal{O}(x)] > 0, \text{ where}$$

$\mathcal{O}(T^r x) = \{T^r x, T^{r+1} x, \ldots\}$, $r \geqq 0$, $T^0 x = x$, then T is said to have diminishing orbital diameters over K (see [2]).

In his paper [7] Kirk proved the following theorem: If T is a nonexpansive mapping of K into itself i.e. $\| Tx - Ty \| \leqq \| x - y \|$, $x, y \in K$, and if K has normal structure, then T has a fixed point in K. This result has been proved for uniformly convex spaces X by Browder [1], Göhde [3] and Goebel [4], the reflexivity of the space and the normal structure of K being consequences of the uniform convexity of X.

Definition 3. ([11]) *A maping T of a subset K of a normed space X into itself is said to have property B_k on K if for every closed subset F of K, mapped into itself by T and containing more than one element, there exist $x \in F$ and $k \in N$ such that* $\| x - T^k x \| < \sup_{y \in F} \| y - T^k y \|$.

1. We are now in a position to formulate our theorems.

Theorem 1. *Let X be a normed space and let T be a mapping of X into itself having the property of f_{RBS-} contraction over X. Then if T has diminishing orbital diameters over X, T has the property B_k over X.*

Proof. Let F be a closed subset of X, mapped into itself by T, containing more than one element. If possible, let, for every element $x \in F$, $\| x - T^k x \| = \sup_{y \in F} \| y - T^k y \| = M$. M is evidently non-zero, for if $M = 0$ then F would contain more than one fixed point of T, which is not possible.

Now we use the following Lemma, proved in our paper [10].

Lemma. ([10], p. 198). *Let $f: R^{k+2} \to R$ $(k \in N)$ be a monotonically increasing (with respect to every real argument) and semihomogenous mapping, let $g(x) = f(\alpha_0, \alpha_1 x, \ldots, \alpha_{k+1} x^{k+1})$ be continuous at the point $x = 1$, and let the sequence (x_n) of nonnegative real numbers satisfy the condition*

$$x_{n+k} < f(\alpha_0 x_n, \alpha_1 x_{n+1}, \ldots, \alpha_k x_{n+k}, \alpha_{k+1} C), \; n \in N,$$

(here k is a fixed natural number, where $\alpha_0, \ldots, \alpha_{k+1}$ and C are nonnegative real constants and $f(\alpha_0, \ldots, \alpha_{k+1}) = \lambda \in [0, 1]$). Then, there exist positive numbers $\mathscr{L}$ and $\theta \in (0, 1]$ such that

$$x_n \leqq \mathscr{L} \theta^n \, (n \in N), \quad \mathscr{L} = \max \{C, \max_{i=1,2,\ldots,k} (x_i \theta^{-i})\}.$$

Especially, for $C = 0$ or $\alpha_{k+1} = 0$ is $\theta \in (0, 1)$.

Corollary of Lemma. *Especially, let $T: X \to X$ be an f_{RBS-}contraction on X and let n be any positive integer. Then for each $x \in X$ and all positive integers i and j*

(a) $1 \leqq i, j \leqq n \Rightarrow \| T^i x - T^j x \| \leqq \lambda \, \partial [\mathcal{O}(x, n)]$, $\mathcal{O}(x, n) = \{x, Tx, \ldots, T^n x\}$,

(b) $(\forall x \in X) \; (\exists k \leqq n) \; \partial [\mathcal{O}(x, n)] = \| x - T^k x \|$,

(c) $\partial [\mathcal{O}(x, \infty)] \leqq (1 - \lambda)^{-1} \| x - Tx \|$, $(\lambda \neq 1)$.

Now, for $x \in F$ and from Lemma and corollary to the Lemma we have

$$\|x_r - x_s\| < f(\alpha_1 \|x_{r-1} - x_{s-1}\|, \alpha_2 \|x_{r-1} - x_r\|, \alpha_3 \|x_{s-1} - x_s\|, \alpha_4 \|x_{r-1} - x_s\|, \alpha_5 \|x_{s-1} - x_r\|) \leqq M,$$

i.e.

$$\|x_r - x_s\| \leqq M; \quad r, s \geqq 1.$$

Hence, for $r \geqq 1$, $\partial[O(x_r)] = \partial(x_r, x_{r+1}, \ldots) = M$, (because $\|x_r - x_s\| \leqq M$ and $\|x_r - x_{r+1}\| = M$). Hence, at $Tx \in F$, T does not have a diminishing orbital diameter. This contradiction completes the proof.

Remark. From corollary to the Lemma we are now in a position to prove our theorem: Let X be a normed space and let T be a mapping of X into itself having the property of f_{RBS}-contraction over X. Then if T has diminishing orbital diameters over X, a mapping T of a subset K of a normed space X into itself and for every closed subset F of K, mapped into itself by T and containing more then one element, there exist $x \in F$ and $k \leqq n$ such that $\|x - T^k x\| < \partial[O(x, n)]$.

In paper [11] we proved the following theorem for Banach space.

Theorem 2. *Let K be a bounded convex subset of a Banach space X and let K be mapped into itself by T. Suppose further that T is an f_{RBS}-contraction on K. Then if K has normal structure, T has the following property: For every closed convex subset F of K mapped into itself by T and containing more than one element there exist an $x \in F$ and $k \in N$ such that $\|x - T^k x\| < \sup_{y \in F} \|y - T^k y\|$.*

2. Main result

We state also the following well-known results of Kannan and Delfina Roux-Paolo Soardy [6], [8]. They were concerned with applications of T, mapping a closed convex subset K of a normed space X into itself, with the property:

$$\text{(Kannan)} \quad \|Tx - Ty\| \leqq 2^{-1}(\|x - Tx\| + \|y - Ty\|); \quad x, y \in K$$

and

$$\text{(D. Roux-P. Soardy)} \quad \|Tx - Ty\| \leqq a(\|x - Tx\| + \|y - Ty\|) + b\|x - y\|$$

$$(a = a(x, y) \geqq 0, \quad b = b(x, y) \geqq 0, \; 2a + b \leqq 1)$$

respectively. If, in addition

1) X is uniformly convex and there exists in K a point with bounded orbit,

2) X is a Banach space, K is weakly compact, T is continuous and

$$\sup\{b(x, y) : x, y \in K\} < 1;$$

they both proved that T has a fixed point in K.

We now prove the following Theorem which is obviously an improvement of the preceding result of Kannan and D. Roux-P. Soardy.

T h e o r e m 3. *Let T be a mapping of a nonempty bounded, closed and convex subset K of a reflexive Banach space X into itself and let T have the property of f_{RBS} — cotraction over K. Then if $(\exists\, k \in N)$ $\sup_{y \in F} \| y - T^k y \| < \partial(F)$ for every nonempty bounded closed convex subset F of K, containing more than one element and mapped into itself by T, T has a unique fixed point in K.*

In the proof of the Theorem we shall make use of the following Theorem.

T h e o r e m 4. ([9], p. 327). *A necessary and sufficient condition that a Banach space X be reflexive is that: Every bounded descending sequence (transfinite) of non-empty closed convex subsets of X has a non-empty intersection.*

P r o o f o f T h e o r e m 3. Let $\mathcal{F}$ be the family of all closed convex bounded subsets of K, mapped into itself by T. Obviously, $\mathcal{F}$ is nonempty. By the result of Smulian [9] and applying Zorn's Lemma, we get a minimal element S in $\mathcal{F}$, S being minimal with respect to being nonempty, bounded, closed and convex and invariant under T. If S contains only one element, then that element is a fixed point of T. If not, let S contain more than one element. Now for $x, y \in S$ (from Corollary to the Lemma)

$$\| Tx - Ty \| < f(\alpha_1 \| x - y \|,\ \alpha_2 \| x - Tx \|,\ \alpha_3 \| y - Ty \|,\ \alpha_4 \| x - Ty \|,\ \alpha_5 \| y - Tx \|)$$

$$\leq \lambda\, \partial [O(x, n)] \leq \sup_{x \in S} \| x - T^k x \|, \quad (k \in N).$$

Hence, $T(S)$ is contained in the closed ball M with T as center and $\sup_{x \in S} \| x - T^k x \|$ as radius. Also $S \cap M$ is invariant under T, therefore, by the minimality of S it follows that $S \subset M$ i.e. $\| Ty - x \| \leq \sup \| x - T^k x \|$, for every $x \in S$. Hence, for any arbitrary but fixed $y \in S$, we have

$$\text{(1)} \qquad \sup_{x \in S} \| Ty - x \| \leq \sup_{x \in S} \| x - T^k x \| .$$

Let $$S' = \{ z \in S : \sup_{x \in S} \| z - x \| \leq \sup_{x \in S} \| x - T^k x \|, \quad k \in N \}.$$

Obviously S' is closed, convex and nonempty ($Ty \in S'$). Again if $z \in S'$, then $z \in S$ and hence $Tz \in S'$ by (1). Hence S' is invariant under T. Also $\partial(S') \leq \sup_{x \in S} \| x - T^k x \| < \partial(S)$, by hypothesis. Hence S' is a proper subset of S, which contradicts the minimality of S. Hence S has only one element which is a fixed point of T. The uniqueness of the fixed point follows from the fact that it $x = Tx$, $y = Ty$ then

$$\| x - y \| = \| Tx - Ty \| \leq \lambda \| x - y \| \Rightarrow x = y,$$

which ends the proof.

Acknowledgement.

I am thankful to prof. Dr. Julian Musielak, Poznań, Poland, for his kind help and suggestions during the preparation of this paper.

REFERENCES

[1] Browder F. E. — *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U. S. A., 54 (1965).

[2] Belluce and W. A. Kirk. — *Fixed point theorems for families of contractions mappings*, Pacif. J. Math. 18 (1966), pp. 213—217.

[3] Göhde D. — *Zum Princip der kontraktiven Abbildung*, Math. Nachr. 30 (1965), p. 251—258.

[4] Goebel K. *An elementary proof of the fixed point theorem of Browder and Kirk*, Michigan Math. J. 16 (1969), 381—383.

[5] Edelstein M. — *On nonexpansive mappings of Banach spaces*, Proc. Cambrigde Philos. Soc. 60 (1964), pp. 439—447.

[6] Kannan R. — *Some results on fixed points — III and IV*, Fund. Math 74 (1972), 181—187, and 70 (1971) pp. 169—177.

[7] Kirk W. A. *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Montly, 72 (1965) pp. 1004—1006.

[8] Roux D. and Paolo Soardy, — *Alcune generalizazioni del Teorema di Browder-Göhde-Kirk*, Lincei-Rend. Sc. fis. mat. e nat., Vol. 52 (1972) pp. 682—688.

[9] Smulian V. *On the principle of inclusion in the space of the type* (*B*), Math. Sb. 5 (47) 1939, pp. 327—328.

[10] Tasković M. — *Einige Abbildungen vom* β*—typus*, Publ. Inst. Math. t. 17 (31), 1973 pp. 198—206.

[11] Tasković M. *Banach mappings in reflexive Banach space* (*to appear*).

M. Tasković
Prirodno-matematički fakultet
11.000 Beograd
p. p. 550
Yugoslavia