PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 20 (34), 1976, pp. 231—242

SOME RESULTS IN THE FIXED POINT THEORY

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(Communicated August 28, 1975)

Abstract. **In this paper we prove a somewhat more general theorems on the convergence of sequences, and prove a fixed point theorem for f-contractive operators.**

1. Introduction

1.0. In the fixed-point theory of contractive operators (mappings which shrinks distance in some manner) on metric spaces is well-known the result of the Polish mathematican S. Banach. Banach's contraction principle can be formulated as follows.

Let $T: X \to X$ be a mapping of a complete metric space (X, ρ) into itself. If T is a *contraction, i.e. if*

(A) $$\rho[Tx, Ty] \leqq \alpha \rho[x, y] \quad \textit{for some} \quad \alpha \in [0, 1),$$

and all $x, y \in X$, *then*

(a) *T has a (unique) fixed point ξ in X,*

(b) $T^n(x) \xrightarrow[n \to \infty]{} \xi$ *for all* $x \in X$,

(c) *There exists an open neighborhood U of ξ such that for any neighborhood V of ξ there is an $n(V)$ which satisfies $n \geqq n(V) \Rightarrow T^n(U) \subset V$.*

In other words, if T is a contractive mapping on a complete metric space X, then the equation $Tx = x$ has in X a unique solution.

In [15] *we introduced the concept of a f-contraction T of a metric space X into itself i.e. of a mapping $T: X \to X$ such that for every $x, y \in X$ there exist nonnegative real numbers $\alpha_i(x, y) = \alpha_i$ $(i = 1, 2, \ldots, 5)$ such that*

(B) $$\rho[Tx, Ty] \leqq f(\alpha_1 \rho[x, y], \alpha_2 \rho[x, Tx], \alpha_3 \rho[y, Ty], \alpha_4 \rho[x, Ty], \alpha_5 \rho[y, Tx]),$$

where $\sup\{f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5): x, y \in X\} = \lambda \in [0, 1)$ *and the mapping $f: R^5 \to R$ is increasing, semihomogeneous and such that* $g(x) \overset{\text{def}}{=} f(\alpha_1 x, \ldots, \alpha_5 x^5)$ *is continuous at the point $x = 1$.*

We say that the mapping $f: R^n \to R (n \in N)$ is semihomogeneous iff $f(\delta x_1, \ldots, \delta x_n) \leqq \delta f(x_1, \ldots, x_n)$, $\delta \geqq 0$. It is clear that the condition of semi-homogeneity implies homogeneity for the mapping f.

We showed that Banach's result for contractions can be extended to f-contractions.

Let T be a mapping of a metric space X into itself. A space X is said to be T-orbitally complete if every Cauchy sequence of the form $\{T^{n_i}x \mid i \in N\}$, $x \in X$, converges in X.

In [15] we have proved the following theorem:

Theorem T. *Let T be a f-contraction on a metric space X and let X be T-orbitally complete, then T has the unique fixed point ξ in X*

Comparing Banach fixed point Theorem with the above Theorem proved in [15] one can see the relationship betwen conditions (A) and (B). It is clear at a first glance that the condition (A) implies the continuity of the map in the whole space but condition (B) does not.

Special cases of (B) have been discussed by Kannan [3], S. Reich [9], Hardy-Rogers [11], Ćirić [5], Đ. Kurepa [4], Fukushime [13], Boyd and Wong [1], Rakotch [7] and others. The following example shows that our theorem is effectively more general than other theorems concerning this case.

Example 1. (c.f. [15]) *Let $X = [0, 1]$ and define $T: X \to X$ by $Tx = 0$ for $x \in [0, 1/2) \cup (1/2, 1]$, $Tx = 2x$, for $x \in \{1/2\}$, and the distance function ρ is the ordinary euclidean distance on the line. The mapping T is a f-contraction given in Example* 2 (*one uses Theorem* 1).

Since X is T-orbitally complete, it follows by Theorem T, *that T has a unique fixed point — it is the point* 0. *On the other hand the conditions of Banach, R. Kannan, S. Reich, Ćirić, Hardy-Rogers and others are not satisfied for $x = 0$ and $y = 1/2$.*

1.1. The purpose of this paper is to consider functions T on X which are not necessarily continuous and which satisfy a condition of the type

$$\text{(C)}\quad \begin{aligned} &(\exists \alpha_i, \beta \in R)(\forall x, y \in X)(\alpha_1 + \alpha_2 + \alpha_3 > \beta \text{ and } (\beta - \alpha_2 \geqq 0 \vee \beta - \alpha_3 \geqq 0)) \\ &\alpha_1 \rho[Tx, Ty] + \alpha_2 \rho[x, Tx] + \alpha_3 \rho[y, Ty] - \min\{\rho[x, Ty], \rho[y, Tx]\} \leqq \beta \rho[x, y]. \end{aligned}$$

Various theorems, about fixed points are obtained.

In [6] Ćirić has considered functions T on X which are not necessarily continuous and which satisfy a condition of the type

$$\text{(D)}\quad \min\{\rho[Tx, Ty], \rho[x, Tx], \rho[y, Ty]\} - \min\{\rho[x, Ty], \rho[y, Tx]\} \leqq \alpha \rho[x, y].$$

In [15] we describe a class of conditions sufficient for the existence of a fixed point, which generalize several known results.

Let T be a mapping of a metric space X into itself. We recall that a mapping T on X is orbitally continuous if $\lim T^{n_i}x = u$ implies $\lim T(T^{n_i}x) = Tu$, for each $x \in X$.

In [6], Ćirić has proved the following theorem:

Theorem Ć. *Let $T: X \to X$ be an orbitally continuous mapping on X and let X be T-orbitally complete. If T satisfies condition* (D), *then for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.*

Comparing theorem Ć with the above Theorem T one can see the relationship betwen conditions (D) and (B). It is clear at a first glance that the conditions of theorem Ć implies the continuity (T-orbitally continuity) of the map in the whole space but conditions of Theorem T does not. On the other hand, the condition (D) implies the condition (B).

1.2. Let ε be positive. A metric space is said to be ε-chainable if for every $p, q \in X$ there exists a finite set of points $p = x_0, x_1, \ldots, q = x_n$ such that $x_i \in X$ and $\rho[x_{i-1}, x_i] < \varepsilon$ for $i = 1, \ldots, n$.

Suppose that (A) holds only for x, y such that $\rho[x, y] < \varepsilon$. If (X, ρ) is a complete ε-chainable metric space this localized condition still implies the existence of a unique fixed point of T (M. Edelstein [2], p. 8). On the other hand, an anologous localized version of (B), is given in [18].

1.3. Let T be a mapping of a metric space X into itself. For $A \subset X$, let $\partial(A) = \sup\{\rho[a, b] \mid a, b \in X\}$, and for each $x \in X$ let

$$\mathcal{O}(T^m x, n) = \{T^m x, \ldots, T^{m+n} x\}, \qquad n = 1, 2, 3, \ldots;$$

$$\mathcal{O}(T^m x, \infty) = \{T^m x, T^{m+1} x, \ldots\}, \quad m = 0, 1, 2, \ldots,$$

where it is understood that $T^0 x = x$.

A space X is said to be T-orbitally complete iff every Cauchy sequence which is contained in $\mathcal{O}(x, \infty)$ for some $x \in X$ converges in X.

Definition. *A mapping $T: X \to X$ is said to be an (ε, f)-contraction iff there exist a positive constant ε such that $\rho[x, y] < \varepsilon$ implies* (B).

Before going to the theorems, we first recollect the following

Proposition. *Let X be a metric space. Then the following are equivalent:*

(a) *X is complete;*

(b) *If S is any nonempty closed subset of X and $T: S \to S$ any f-contraction, then T has a fixed point.*

This is an immediate corollary od theorem T and of a result of HU [12].

2. Some specially single-valued maps which are *f*-contraction

Now we can prove our main results.

Theorem 1. *Let $T: X \to X$ be a mapping on X and let X be a T-orbitally complete metric space. If T satisfies the following condition: there exist real numbers α_i, β for every $x, y \in X$ such that: $\alpha_1 + \alpha_2 + \alpha_3 > \beta$ and $\beta - \alpha_2 \geqq 0 \vee \beta - \alpha_3 \geqq 0$, and*

$$(1) \quad \alpha_1 \rho[Tx, Ty] + \alpha_2 \rho[x, Tx] + \alpha_3 \rho[y, Ty] - \min\{\rho[x, Ty], \rho[y, Tx]\} \leqq \beta \rho[x, y],$$

then for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.

Proof. Let $x \in X$ be arbitrary. We shall show that the sequence of iterates

$$(2) \qquad x_0 = x, \ x_n = T(x_{n-1}); \quad n = 1, 2, 3, \ldots ;$$

at x is a Cauchy sequence. Since $x_{k-1} = x_k$ for some $k \in N$ immediately implies that $\{x_n\}$ is the Cauchy's sequence, we can suppose that $x_{n-1} \neq x_n$ for each $n = 1, 2, \ldots$. By (1) for $x = x_{n-1}$ and $y = x_n$ we have

$$\alpha_1 \rho[x_n, x_{n+1}] + \alpha_2 \rho[x_{n-1}, x_n] + \alpha_3 \rho[x_n, x_{n+1}] - \min\{\rho[x_{n-1}, x_{n+1}], 0\}$$
$$= (\alpha_1 + \alpha_3)\rho[x_n, x_{n+1}] + \alpha_2 \rho[x_{n-1}, x_n] \leqq \beta\rho[x_{n-1}, x_n]$$

i.e.

$$\rho[x_n, x_{n+1}] \leqq \left(\frac{\beta - \alpha_2}{\alpha_1 + \alpha_3}\right) \rho[x_{n-1}, x_n].$$

Proceeding in this manner we obtain

$$\rho[x_n, x_{n+1}] \leqq \left(\frac{\beta - \alpha_2}{\alpha_1 + \alpha_3}\right) \rho[x_{n-1}, x_n] \leqq \cdots \leqq \left(\frac{\beta - \alpha_2}{\alpha_1 + \alpha_3}\right)^n \rho[x, Tx].$$

Hence for any $s \in N$ one has

$$\rho[x_n, x_{n+s}] \leqq \sum_{i=1}^{n+s-1} \rho[x_i, x_{i+1}] \leqq \left(\frac{\beta - \alpha_2}{\alpha_1 + \alpha_3}\right)^n \left(1 - \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3}\right)^{-1} \rho[x, Tx].$$

Since $\lim_n (\beta - \alpha_2)^n (\alpha_1 + \alpha_3)^{-n} = 0$, it follows that (2) is a Cauchy sequence. X being T-orbitally complete, there is some $\xi \in X$ such that $\xi = \lim_n T^n x$. To prove $T\xi = \xi$, consider the following inequalities, for $x \equiv T^n x$, and $y = \xi$:

$$\alpha_1 \rho[T^{n+1}x, T\xi] + \alpha_2 \rho[T^n x, T^{n+1}x] + \alpha_3 \rho[\xi, T\xi] - \min\{\rho[T^n x, T\xi],$$
$$\rho[T^{n+1}x, \xi]\} \leqq \beta\rho[T^n x, \xi].$$

Hence, letting n tend to infinity, it follows $\rho[\xi, T\xi] = 0$, i.e. $T\xi = \xi$, which concludes the proof.

This proof is made under the assumption that $\beta - \alpha_2 \geqq 0 \ (\Rightarrow \alpha_1 + \alpha_3 > 0)$. We can also prove the Theorem when $\beta - \alpha_3 \geqq 0 \ (\Rightarrow \alpha_1 + \alpha_2 > 0)$ in a similar way, using the fact that distance is a symmetric function.

Since condition (D) implies the condition (1), our Theorem 1 is a generalization of Theorem Ć of [6]. The following, example shows that (D) does not imply (1).

Example 2. *Let* $X = [0, 2]$, *and define* $T: X \to X$, *by* $Tx = 0 \ (0 \leqq x < 1)$, $Tx = 10/11\, x \ (1 \leqq x \leqq 2)$; *the distance function* ρ *is the ordinary euclidean distance on the line. Then* T *is orbitally continuous, satisfies condition* (1) *of Theorem* 1, *with* $\alpha_1 = -1$, $\alpha_3 = 2$, $\alpha_2 = 0$, $\beta = 2/3$ *and since* X *is* T*-orbitally complete, it follows by Theorem* 1 *that* T *has a fixed point vid. the point* 0. *To show that* T *does not satisfy condition* (D) *in Theorem* Ć, $x = 10/11$ *and* $y = 1$. *Then* $\min\{\rho[Tx, Ty], \ \rho[x, Tx], \ \rho[y, Ty]\} - \min\{\rho[x, Ty], \ \rho[y, Tx]\} = \min\{10/11, 10/11, 1/11\{ - \min\{0, 1\} = 1/11 > \alpha\, 1/11 = \alpha\rho[x, y]$, *and we see that condition* (D) *is not satisfied.*

Theorem 2. *Let $T: X \to X$ be an orbitally continuous mapping on a metric space X which satisfies the following condition*

(3) $\alpha_1 \rho[Tx, Ty] + \alpha_2 \rho[x, Tx] + \alpha_3 \rho[y, Ty] - \min\{\rho[x, Ty], \rho[y, Tx]\} < \beta\rho[x, y]$

whenever $x \neq y$ and $\alpha_1 + \alpha_2 + \alpha_3 \geqq \beta$ and $\beta - \alpha_2 > 0 \vee \beta - \alpha_3 > 0$ (α_i, β are real constants). If for some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a cluster point $\xi \in X$, then ξ is a fixed point of T.

Proof. If $T^{r-1} x_0 = T^r x_0$ for some $r \in N$, then $T^n x_0 = T^r x_0 = \xi$ for all $n \geqq r$, and the assertion follows. Assume now that $T^{r-1} x_0 \neq T^r x_0$ for all $r \in N$, and let $\lim_i T^{n_i} x_0 = \xi$. Then for $T^{n-1} x_0$, $T^n x_0 \in X$, by (3)

$$\alpha_1 \rho[T^n x_0, T^{n+1} x_0] + \alpha_2 \rho[T^{n-1} x_0, T^n x_0] + \alpha_3 \rho[T^n x_0, T^{n+1} x_0] - \min\{\rho[T^{n-1} x_0, T^{n+1} x_0], 0\} = (\alpha_1 + \alpha_3)\rho[T^n x_0, T^{n+1} x_0] + \alpha_2 \rho[T^{n-1} x_0, T^n x_0] < \beta\rho[T^{n-1} x_0, T^n x_0]$$

i.e.

$$\rho[T^n x_0, T^{n+1} x_0] < \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3} \rho[T^{n-1} x_0, T^n x_0] \leqq \rho[T^{n-1} x_0, T^n x_0].$$

Hence

$$\rho[T^n x_0, T^{n+1} x_0] < \rho[T^{n-1} x_0, T^n x_0].$$

Therefore, $\{\rho[T^n x_0, T^{n+1} x_0]\}$ is a decreasing and hence convergent sequence of positive real numbers. Since

$$\lim_i \rho[T^{n_i} x_0, T^{n_i+1} x_0] = \rho[\xi, T\xi] \text{ and } \{\rho[T^{n_i} x_0, T^{n_i+1} x_0]\} \subseteq \{\rho[T^n x_0, T^{n+1} x_0]\},$$

it follows that

$$\lim_n \rho[T^n x_0, T^{n+1} x_0] = \rho[\xi, T\xi]. \tag{4}$$

Also, as $\lim_i T^{n_i+1} x_0 = T\xi$, $\lim_i T^{n_i+2} x_0 = T^2\xi$ and

$$\{\rho[T^{n_i+1} x_0, T^{n_i+2} x_0]\} \subseteq \{\rho[T^n x_0, T^{n+1} x_0]\},$$

by (4)

$$\rho[T\xi, T^2\xi] = \rho[\xi, T\xi]. \tag{5}$$

Suppose that $\rho[\xi, T\xi] > 0$. Then by (3) we have

$$\rho[T\xi, T^2\xi] < \rho[\xi, T\xi],$$

which contradicts (5). This proves that $T\xi = \xi$.

This proof is made under the assumption that $\beta - \alpha_2 > 0$ ($\Rightarrow \alpha_1 + \alpha_3 > 0$). We can also prove the Theorem when $\beta - \alpha_3 > 0$ ($\Rightarrow \alpha_1 + \alpha_2 > 0$) in a similar way, using the fact that distance is a symmetric function.

Since the condition (D) implies (3), Theorem 2 is a generalization of Theorem 3 of [6].

3. On the convergence of certain sequences

In [8] and [16] some theorems on the convergence of certain sequences were proved. In this section we prove a somewhat more general theorem on the convergence of sequences and give a number of examples and corollaries.

Proposition 1. *Let* $f: R^{k+1} \to R$ $(k \in N)$ *be monotonically increasing (with respect to every real argument) and semihomogeneous mapping, let* $g(x) \overset{\text{def}}{=} f(\alpha_0, \alpha_1 x, \ldots, \alpha_k x^k)$ *be continuous at the point* $x=1$, *and let the sequence* $\{x_n\}$ *of nonnegative real numbers satisfy the condition*

$$x_{n+k} \leqq f(\alpha_0 x_n, \alpha_1 x_{n+1}, \ldots, \alpha_k x_{n+k}), \quad n \in N; \tag{6}$$

(k fixed natural number), where $\alpha_0, \alpha_1, \ldots, \alpha_k$ *are nonnegative real constants and* $f(\alpha_0, \alpha_1, \ldots, \alpha_k) \in [0, 1)$. *Then, there exist numbers* $\mathscr{L} > 0$ *and* $\theta \in [0, 1)$ *such that*:

$$x_n \leqq \mathscr{L} \theta^n \quad (n=1, 2, \ldots). \tag{7}$$

Proof. Since the mapping $g(x)$ is continuous at the point $x=1$ and $g(1)<1$, the function $F(x)=x^{-k} g(x)$ is continuous at the point $x=1$ and $F(1)<1$. It follows that there exists $\theta \in [0, 1)$ such that $F(\theta)<1$, i.e.

$$g(\theta) < \theta^k. \tag{8}$$

Put $\mathscr{L} = \max\limits_{i=1, \ldots, k} (x_i \theta^{-i})$. Then we have

$$x_i \leqq \mathscr{L} \theta^i \quad (i=1, 2, \ldots, k),$$

i.e. (7) holds for $n=1, 2, \ldots, k$. Let us suppose that (7) holds from 1 to $n+k-1$, i.e. that

$$x_i \leqq \mathscr{L} \theta^i \quad (i=1, 2, \ldots, n+k-1).$$

Then we have, by (8)

$$\begin{aligned} x_{n+k} &\leqq f(\alpha_0 x_n, \alpha_1 x_{n+1}, \ldots, \alpha_k x_{n+k}) \\ &\leqq \mathscr{L} \theta^n g(\theta) \leqq \mathscr{L} \theta^{n+k}. \end{aligned}$$

So (7) holds for every $n \in N$.

Remarks 1). If we substitute the condition (6) by

$$x_{n+k} < f(\alpha_0 x_n, \alpha_1 x_{n+1}, \ldots, \alpha_k x_{n+k}, \alpha_{k+1} M), \quad n \in N;$$

($f: R^{k+2} \to R$, k fixed natural number), where $\alpha_0, \alpha_1, \ldots, \alpha_{k+1}, M$ are nonnegative real constants and $f(\alpha_0, \alpha_1, \ldots, \alpha_{k+1}) \in [0, 1]$, then Proposition 1 still holds with $\theta \in [0, 1]$ and $\mathscr{L} = \max\{M, \max\limits_{i=1, \ldots, k} (x_i \theta^{-i})\}$.

The proof is similar to the proof of Proposition 1.

2). When the semihomogeneity in Proposition is substituted by $f(\delta x_1, \ldots, \delta x_k) \leqq \delta f(x_1, \ldots, x_k)$, $\delta \in [\alpha, \infty) \subset R_+$, then also are valid the conditions of the Proposition 1 beside other suppositions. The proof is similar, only that the element $\mathscr{L} \in R_+$ is

$$\mathscr{L} = \max_{i=1, 2, \ldots, k} (\alpha, x_i \theta^{-i}).$$

We state some corollaries and applications of Proposition 1:

L e m m a 1. (Corollary of the Proposition 1, see [15]). *Let* $T: X \to X$ *be a f-contraction on* X *and let* n *be any positive integer. Then for each* $x \in X$ *and all positive integers* i *and* j

(a) $1 \leqq i, j \leqq n \Rightarrow \rho[T^i x, T^j x] \leqq \lambda \partial[\mathcal{O}(x, n)]$;

(b) $(\forall x \in X)(\exists k \leqq n)\, \rho[x, T^k x] = \partial[\mathcal{O}(x, n)]$;

(c) $\partial[\mathcal{O}(x, \infty)] \leqq (1-\lambda)^{-1} \rho[x, Tx]$.

C o r o l l a r y 1 (Prešić [8], p. 76).

Let $\{x_n\}$ *be a sequence of nonnegative numbers satisfying the condition*

$$x_{n+k} \leqq \alpha_1 x_n + \cdots + \alpha_k x_{n+k-1} \quad (n = 1, 2, \ldots,),$$

$\alpha_1, \alpha_2, \ldots, \alpha_k$ *being nonnegative constants. Then there exists positive numbers* $\mathcal{L}$ *and* θ *such that* (7) *holds.*

C o r o l l a r y 2. (Đ. Kurepa [4], p. 103) *If we have in a metric space*

$$\rho[x_{n+1}, x_n] \leqq \alpha_1 \rho[x_n, x_{n-1}] + \cdots + \alpha_k \rho[x_{n-k-1}, x_{n-k}] \quad (\alpha_i \geqq 0),$$

where $\alpha_1 + \cdots + \alpha_k \in [0, 1)$, *then the sequence* $\{x_n\}$ *is fundamental.*

Applying this Proposition, we proved in [18] the following result.

T h e o r e m T 1 ([18], p. 199). *Let* (X, ρ) *be a complete metric space and let* T *be a mapping of* X^k *to* X *satisfy the condition*

$$\rho[T(u_1, \ldots, u_k), T(u_2, \ldots, u_{k+1})] \leqq f\Big(\alpha_1 \rho[u_k, T(\bar{u}_1)],\ \alpha_2 \rho[u_{k+1}, T(\bar{u}_2)],$$

$$\alpha_3/2\, \rho[u_k, T(\bar{u}_2)],\ \alpha_4/2\, \rho[u_{k+1}, T(\bar{u}_1)],\ \sum_{i=1}^{k} q_i \rho[u_i, u_{i+1}]\Big),$$

for every $u_1, \ldots, u_{k+1} \in X$; $\alpha_i, q_i \in R_+$ *where* $f\left(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \sum_{i=1}^{k} q_i\right) \in [0, 1)$, $T(\bar{u}_i) \overset{\text{def}}{=} T(u_i, u_{i+1}, \ldots, u_{i+k-1})$, *and let the mapping* $f: R^5 \to R$ *be increasing, semihomogeneous and* $g(x) = f(\alpha_1 x, \alpha_2 x^2, \ldots, \Sigma q_i x^5)$ *continuous at the point* $x = 1$. *Then*

(a) *There exists a fixed point* $\xi \in X$ *of mapping* $\mathcal{T}(x) \overset{\text{def}}{=} T(x, \ldots, x)$.

(b) ξ *is the limit of the sequence* $\{x_n\}$ *satisfying*

$$x_{n+k} = T(x_n, \ldots, x_{n+k-1}), \quad n \in N;$$

independently of initial values.

(c) *The rapidity of convergence of the sequence* $\{x_n\}$ *to the point* ξ *is evaluated by*

$$\rho[x_{n+k}, \xi] \leqq \theta^n (1-\theta)^{-1} \max_{i=1, \ldots, k} (\rho[x_i, x_{i+1}]\, \theta^{-i}), \quad (\theta \in (0, 1),\ n \in N).$$

4. A localization theorem

In this section we prove a somewhat more general theorem on fixed points and give a number of examples. In Prešić's paper [8] a theorem on fixed points has been proved.

Let $f: R^k \to R$ $(k \in N)$ be increasing, semihomogeneous mapping, let $g(x) = f(\alpha_1 x, \ldots, \alpha_k x^k)$ be continuous at the point $x = 1$, where $\alpha_i (i = 1, 2, \ldots, k)$ are nonnegative real constants. In this section all mappings of the above type will be referred to as having the property (F).

Theorem 3. *Let (X, ρ) be a complete metric space and let T be a mapping of X^k to X satisfying the condition*

(E) $\rho[T(u_1, \ldots, u_k), T(u_2, \ldots, u_{k+1})] \leqq f(\alpha_1 \rho[u_1, u_2], \ldots, \alpha_k \rho[u_k, u_{k+1}])$

for every $u_1, u_2, \ldots, u_{k+1} \in X$, where $f(\alpha_1, \ldots, \alpha_k) \in [0, 1)$ and the mapping $f: R^k \to R$ (k given natural number) has the property (F).

Then:

(a) *There exists a fixed point $\xi \in X$ of the mapping $\mathcal{T}(x) \overset{\text{def}}{=} T(x, \ldots, x)$, and it is unique when $f(\alpha_1, 0, \ldots, 0) + \cdots + f(0, \ldots, 0, \alpha_k) < 1$.*

(b). *ξ is the limit of the sequence $\{x_n\}$ satisfying*

(9) $$x_{n+k} = T(x_n, \ldots, x_{n+k-1}), \quad n \in N;$$

independently of initial values.

(c) *The rapidity of convergence of the sequence $\{x_n\}$ to the point ξ is evaluated by*

$$\rho[x_{n+k}, \xi] \leqq \theta^n (1-\theta)^{-1} \max_{i=1,\ldots,k} (\rho[x_i, x_{i+1}]\theta^{-i}); \quad \theta \in (0, 1), \quad n \in N.$$

Proof. We show first that (9) is a Cauchy sequence. We have, by (F)

$$\rho[x_{n+k}, x_{n+k+1}] = \rho[T(x_n, \ldots, x_{n+k-1}), T(x_{n+1}, \ldots, x_{n+k})]$$
$$\leqq f(\alpha_1 \rho[x_n, x_{n+1}], \ldots, \alpha_k \rho[x_{n+k-1}, x_{n+k}]).$$

Applying Proposition 1 to the sequence $\{\rho[x_n, x_{n+1}]\}$ we obtain, accordingly to (7)

$$\rho[x_n, x_{n+1}] \leqq \theta^n \max_{i=1,\ldots,k} (\rho[x_i, x_{i+1}]\theta^{-i}), \quad \theta \in (0, 1), \quad n \in N.$$

Hence for $n, s \in N$.

(10) $$\rho[x_n, x_{n+s}] \leqq \sum_{j=1}^{s} \rho[x_{n+j-1}, x_{n+j}]$$
$$\leqq \max_{i=1,\ldots,k} (\rho[x_i, x_{i+1}]\theta^{-1}) \sum_{j=1}^{s} \theta^{n+j-1}$$
$$\leqq \theta^n (1-\theta)^{-1} \max_{i=1,\ldots,k} (\rho[x_i, x_{i+1}]\theta^{-1}),$$

which implies that $\{x_n\}$ is a Cauchy sequence. Hence, the metric space X being complete, there exists $\xi = \lim_n x_n$.

Let us prove that ξ is a fixed point of T (in the sense precised above). To show that $T(\xi, \ldots, \xi)=\xi$, where $\xi=\lim_n x_{n+k}$, let us denote

$$x^i_{n+k}=T(x_{n+i}, x_{n+i+1}, \ldots, x_{n+k-1}, \xi, \ldots, \xi)$$

for $n=1, 2, 3, \ldots$; $i=0, 1, \ldots, k$. Note that, in particular,

$$x^0_{n+k}=T(x_n, x_{n+1}, \ldots, x_{n+k-1})=x_{n+k}, \quad x^k_{n+k}=T(\xi, \ldots, \xi).$$

Then for $n=1, 2, \ldots$, we have

$$\rho[\xi, T(\xi, \ldots, \xi)] \leqq \rho[\xi, x^0_{n+k}]+\rho[x^0_{n+k}, x^1_{n+k}]+\cdots+\rho[x^{k-1}_{n+k}, x^k_{n+k}],$$

by the triangle inequality. Using (E) and then the triangle inequality, if we denote

$$\Delta^j=\Delta^j_n=\rho[x^j_{n+k}, x^{j+1}_{n+k}], \quad (j=0, 1, 2, \ldots, k-1),$$

we obtain

$$\Delta^j \leqq f(\alpha_1 \rho[x_{n+j}, x_{n+j+1}], \ldots, \alpha_k \rho[x_{n+k-1+j(k)}, \xi])$$

Hence, from Proposition 1 (letting $n \to \infty$), we get

$$\xi=\lim_n x_{n+k}=T(\xi, \ldots, \xi),$$

which was to be proved. If ξ^* is an element of X such that $T(\xi^*, \ldots, \xi^*)=\xi^*$ then

$$\begin{aligned}
&\rho[\xi, \xi^*]=\rho[T(\xi, \ldots, \xi), T(\xi^*, \ldots, \xi^*)]\\
&\leqq \rho[T(\xi, \ldots, \xi), T(\xi, \ldots, \xi, \xi^*)]+\\
&\rho[T(\xi, \ldots, \xi, \xi^*), T(\xi, \ldots, \xi, \xi^*, \xi^*)]+\cdots\\
&+\rho[T(\xi, \xi^*, \ldots, \xi^*), T(\xi^*, \xi^*, \ldots, \xi^*)] \leqq\\
&\leqq \rho[\xi, \xi^*]\{f(\alpha_1, 0, \ldots, 0)+f(0, \alpha_2, 0, \ldots)+\cdots+f(0, \ldots, 0, \alpha_k)\}\\
&<\rho[\xi, \xi^*].
\end{aligned}$$

This contradiction proves our assertion. Making $s \to \infty$ in (10), one gets (c). The proof is complete.

Corollary 1. (S. Prešić [8]) *Let (X, ρ) be a complete metric space and $T: X^k \to X$ $(k \in N)$ a mapping satisfying the condition*

$$\rho[T(u_1, u_2, \ldots, u_k), T(u_2, \ldots, u_{k+1})] \leqq q_1 \rho[u_1, u_2]+\cdots+q_k \rho[u_k, u_{k+1}]$$

$$(u_1, u_2, \ldots, u_{k+1} \in X; \ \sum q_i<1; \ q_i \geqq 0)$$

Then every sequence $\{x_n\}$ satisfying the condition (9) *is convergent, and* $\lim x_n$ *is the unique solution of the equation* $x=T(x, \ldots, x)$.

Proposition 2. *Let $T: R^k \to R$ satisfy the following condition*:

$$|T(u_1, \ldots, u_k)-T(u_2, \ldots, u_{k+1})| \leqq f(\alpha_1|u_1-u_2|, \ldots, \alpha_k|u_k-u_{k+1}|)$$

for every $u_1, u_2, \ldots, u_{k+1} \in R$, $f(\alpha_1, \ldots, \alpha_k) \in [0, 1)$, where the mapping $f: R^k \to R$ satisfies property (F). *Then every sequence $\{x_n\}$ satisfying the condition* (9) *is convergent and* $\lim_n x_n$ *is the solution of the equation* $x=T(x, \ldots, x)$.

5. Some results of *f*-contractions

There exists a local form of Banach's fixed point theorem (c.f. [18], . 203). Its analogue is.

Theorem 4. *Let $x_0 \in X$ and let*

$$B = B(x_0, r) \stackrel{\text{def}}{=} \{x \in X \mid \rho[x_0, x] \leqq r\},$$

where X is a metric space. If $T: B \to X$ is an f-contraction on B, X is T-orbitally complete and $\rho[x_0, Tx_0] \leqq (1-\lambda) r$, then

(a) *T has in B a unique fixed point ξ,*

(b) *ξ is a limit point of the sequence*

(11) $$x_0,\ x_n = T(x_{n-1}), \quad n \in N;$$

and

(c) $\rho[T^n x_0, \xi] \leqq \lambda^n r \quad (n \in N)$.

Proof. Since $\lambda \in [0, 1)$, from condition $\rho[x_0, Tx_0] \leqq (1-\lambda) r$ it tollows that $x_1 \in B$. By induction we shall show that this sequence is contained in B. Suppose $x_0, x_1, \ldots, x_m \in B$. Then we can apply Proposition and Lemma 1, for $n = 1, 2, \ldots, m$. Thus from Lemma 1 follows

$$\rho[x_0, x_{m+1}] \leqq (1-\lambda)^{-1} \rho[x_0, Tx_0],$$

and by $\rho[x_0, Tx_0] \leqq (1-\lambda) r$ we have $\rho[x_0, x_{m+1}] \leqq r$, i.e. $x_{m+1} \in B$. So the sequence $\{T^n x_0 \mid n \in N\}$ is contained in B. By the same procedures already used, and by routine calculation one can show that (11) is a Cauchy sequence, and has a limit point $\xi \in X$, which must be a fixed point under T. Since B is closed, ξ is in B and so the theorem is proved.

Theorem 5. *Let T be an (ε, f)-contraction of a T-orbitally complete metric space X into itself. Then for every $x \in X$ the following alternative holds: Either*

(I) *for every integer $s = 0, 1, 2, \ldots$, one has $\rho[T^s x, T^{s+1} x] \geqq \varepsilon$, or*

(II) *the sequence $\{T^n x \mid n \in N\}$ converges to a fixed point under T.*

Proof. Let $x \in X$ and consider the sequence of numbers $\{\rho[T^s x, T^{s+1} x] \mid s = 0, 1, 2, \ldots\}$. There are two possibilities: Either

(a) for every integer $s = 0, 1, \ldots$, one has $\rho[T^s x, T^{s+1} x] \geqq \varepsilon$, which is precisely the alternative (I) of the conclusion of the theorem, or else

(b) for some integer $s = s_0$, one has $\rho[T^{s_0} x, T^{s_0+1} x] < \varepsilon$. The proof will be complete if we show that (b) implies alternative (II) of the conclusion of the theorem. Since T is an (ε, f)-contraction and $\rho[T^{s_0} x, T^{s_0+1} x] < \varepsilon$, then repeating the procedure used and routine calculation we obtain from Lemma 1 for $T^{s_0+1} x$ and $T^{s_0+2} x$ that

$$\rho[T^{s_0+1} x, T^{s_0+2} x] \leqq \lambda\, \partial[\mathcal{O}(T^{s_0} x, 1)] \leqq \lambda \varepsilon < \varepsilon,$$

where $\lambda=\sup\{f(\alpha_1, \ldots, \alpha_5)\,|\,x, y\in X\}$ and $\rho[x, y]<\varepsilon$. By induction it follows that

$$\rho[T^{s_0+p}x,\ T^{s_0+p+1}x]\leqq\lambda^p\partial[O(T^{s_0}x,\ 1)]<\varepsilon$$

for every integer $p=0, 1, 2, \ldots,$ and hence, we have (from Lemma 1) for $n>s_0$

$$\rho[T^n x,\ T^{n+p}x]\leqq\lambda^{n-s_0}(1-\lambda)^{-1}\rho[T^{s_0}x,\ T^{s_0+1}x].$$

Thus the sequence, $\{T^n x\,|\,n=0, 1, 2, \ldots\}$ is a Cauchy sequence and let $\xi\in X$ be such that $\xi=\lim_n T^n x=\lim_p T^{s+p}x$. Since $\lambda<1$ and

$$\rho[T^n x,\ \xi]\leqq\lambda^{n-s_0}(1-\lambda)^{-1}\rho[T^{s_0}x,\ T^{s_0+1}x]$$

for $n>s_0$, it follows that there is an integer $n_0>s_0$ such that $\rho[T^n x, \xi]<\varepsilon$ for every $n>n_0$. Then, from Proposition 1 for all $n>n_0>s_0$, we have

$$\rho[T\xi,\ T(T^n x)]\leqq f(\alpha_1\rho[\xi,\ T^n x],\ \alpha_2\rho[\xi,\ T\xi],\ \alpha_3\rho[T^n x,\ T^{n+1}x],$$
$$\alpha_4\rho[\xi,\ T^{n+1}x],\ \alpha_5\rho[T^n x,\ T\xi])\leqq\mathscr{L}\,\theta^n\quad(n\in N,\ \theta\in(0, 1)).$$

Hence, letting n tend to infinity, it follows $\rho[T\xi, \xi]=0$, i.e. $T\xi=\xi$, which concludes the proof.

The idea of the proof is due to M. Margolis — Diaz [14] and Lj. Ćirić [5].

Edelstein ([2], p. 7) has proved another localized version of Banach's theorem for a complete ε-chainable metric space, namely

Theorem E. *Let T be a mapping of a complete ε-chainable metric space (X, ρ) into itself such that*

$$\rho[x, y]<\varepsilon\ \Rightarrow\ \rho[Tx, Ty]\leqq\alpha\rho[x, y],\quad\alpha\in[0, 1).$$

Then T has a unique fixed point.

As a special case of the above theorem 3 we have the following result:

Proposition 3. *Let $T:X\to X$ be an (ε, f)-contraction on a T-orbitally complete metric space X into itself. If for every $x\in X$ there exists an integer $n(x)$ such that $\rho[T^{n(x)}x,\ T^{n(x)+1}x]<\varepsilon$ and if p and q are two fixed points of T such that $\rho[p, q]<\varepsilon$, then T has exactly one fixed point, and all sequences $\{T^n x\,|\,n\in N\}$ converge to the unique fixed point of T.*

Remark. In [5] Lj. Ćirić has introduced (ε, λ)-uniformly locally generalized contraction by requiring the existence of a positive constant ε such that

$$(\acute{C})\quad \rho[x, y]<\varepsilon\Rightarrow\rho[Tx, Ty]\leqq q\rho[x, y]+r\rho[x, Tx]+s\rho[y, Ty]+t(\rho[x, Ty]+\rho[y, Tx])$$

when $\sup_{x, y\in X}\{q+r+s+2t\}<1,\ (q, r, s, t\geqq 0)$.

Since the condition (Ć) implies the condition of an (ε, f)-contraction, Theorem 5 is a generalizations of Theorem 4 of [5].

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