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BALANCED LAWS ON TERNARY *GD*-GROUPOIDS

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In this paper the results, which we obtained in [7], are generalized. Ternary *GD* groupoids which satisfy balanced laws of the first kind are examined. It is shown that the theorems from [7] can be generalized, under some additional assumptions, for ternary *GD* groupoids.

Definitions of the basic notions and notation which we use are given in [7] (similar notation is used in [1], [2], [3], [4]). Other notions from the theory of n-ary quasigroups can be found in [5] and about *GD*-groupoids in [3] and [6].

Let $w_1 = w_2$ be a balanced law of the first kind in which all operations are ternary and let the set of all operation letters from the term w_1 be $\Phi_1 = \{A_1, \ldots, A_n\}$ and the set of all operation letters from w_2 be $\Phi_2 = \{B_1, B_2, \ldots, B_n\}$. Let $[w_1] = [w_2] = \{x_1, x_2, \ldots, x_{2n+1}\}$ be the set of all variables appearing in terms w_1 and w_2, and let A_1 and B_1 be the initial operation letters of the terms w_1 and w_2 respectively. Let $S_1, S_2, \ldots, S_{2n+1}, P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_n$ be nonempty sets and $P_1 = Q_1 = S$.

We define ternary *GD*-groupoids A_i, B_i $(i = 1, 2, \ldots, n)$ in the following way. If $A_k(u_1, u_2, u_3)$ is a subterm of the term w_1 then A_k is ternary *GD*-groupoid which maps

$$A_k : M_1 \times M_2 \times M_3 \to P_k,$$

where

$$M_j = S_t \text{ if } u_j = x_t,$$

$$M_j = P_t \text{ if } u_j = A_t(u_1', u_2', u_3'); \qquad j = 1, 2, 3.$$

If $B_k(v_1, v_2, v_3)$ is a subterm of the term w_2 then B_k is ternary *GD*-groupoid which maps

$$B_k : M_1 \times M_2 \times M_3 \to Q_k,$$

where

$$M_j = S_t \text{ if } v_j = x_t,$$

$$M_j = Q_t \text{ if } v_j = B_t(v_1', v_2', v_3'); \qquad j = 1, 2, 3.$$

Let these ternary *GD*-groupoids A_i, B_i $(i = 1, 2, \ldots, n)$ satisfy the balanced law of the first kind $w_1 = w_2$. If $a_1, a_2, \ldots, a_{2n+1}$ are arbitrary fixed elements from the sets $S_1, S_2, \ldots, S_{2n+1}$ respectively, $A(u_1, u_2, u_3)$ arbitrary subterm of the terms w_1 or w_2, then mappings L_j^A, L_{jk}^A, σ_A we define analogously as in [7].

Here the mappings L_j^A and σ_A will be surjections and L_{jk}^A binary GD-groupoid which we call the derived binary GD-groupoid from the ternary GD-groupoid A.

If $\overline{\Phi}_i$ denotes the set of all derived binary operations from all ternary operations from $\Phi_i (i=1, 2)$, then the relation "$<$" in the sets $\overline{\Phi}_i \cup [w_i]$ $(i=1, 2)$ and the equivalence relation "$\sim$" in the set $\overline{\Phi} = \overline{\Phi}_1 \cup \overline{\Phi}_2$ we define as in [7].

Then we have

Lemma 1. *Let $w_1 = w_2$ be a balanced law of the first kind in which all operations are ternary GD-groupoids. Let the set $\overline{\Phi}$ of all derived binary GD-groupoids be partitioned into equivalence classes by the relation $\sim$: $\overline{\Phi} = K_1 \cup K_2 \cup \cdots \cup K_p$. If K_i is equivalence class such that K_i contains two connected* ([7]) *binary GD-groupoids L_{jk}^A and L_{lm}^B with the property that among the following four pairs of mappings*

$$\begin{array}{lll} 1) & \sigma_A L_j^A, & \sigma_A L_k^A, \\ 2) & \sigma_B L_l^B, & \sigma_B L_m^B, \\ 3) & \sigma_A L_j^A, & \sigma_B L_m^B, \\ 4) & \sigma_A L_k^A, & \sigma_B L_l^B, \end{array}$$

at least in one pair both mappings are bijections for every choice of elements $a_1, a_2, \ldots, a_{2n+1}$ from the sets $S_1, S_2, \ldots, S_{2n+1}$ respectively, then there exists binary loop $\boldsymbol{i}$, defined on the set S, which is the homotopic image of all derived binary GD-groupoids from the class K_i.

Proof. We define the binary operation $\boldsymbol{i}$ on the set S, if in one of the pairs 1, 3, 4 both mappings are bijections, by

$$\sigma_A L_{jk}^A (x, y) = \sigma_A L_j^A x \, \boldsymbol{i} \, \sigma_A L_k^A y,$$

and, if the mappings in 2 are bijections, by

$$\sigma_B L_{lm}^B (x, y) = \sigma_B L_l^B x \, \boldsymbol{i} \, \sigma_B L_m^B y.$$

It is obvious that the operation $\boldsymbol{i}$ is well defined in the case 1 and 2. Now we shall assume that the mappings from 3 are bijections and show that the operation $\boldsymbol{i}$ is well defined in this case also.

Let s and t be arbitrary elements from the set S. The mapping $\sigma_A L_j^A$ is a bijection and there exists one and only one element x such that $s = \sigma_A L_j^A x$, the mapping $\sigma_A L_k^A$ is a surjection and it is possible that there exist two different elements y_1 and y_2 such that $t = \sigma_A L_k^A y_1 = \sigma_A L_k^A y_2$. Since the operations L_{jk}^A and L_{lm}^B are connected there exist variables x_q and x_r such that, fixing in the law $w_1 = w_2$ all variables, except x_q and x_r, we have

$$\sigma_A L_{jk}^A (\alpha x_q, \beta x_r) = \sigma_B L_{lm}^B (\gamma x_q, \delta x_r),$$

where $\alpha, \beta, \gamma, \delta$ are surjections. From the preceding equation (fixing first x_q and then x_r) we get

$$\sigma_A L_j^A \alpha = \sigma_B L_l^B \gamma,$$

$$\sigma_A L_k^A \beta = \sigma_B L_m^B \delta. \tag{1}$$

Since β is a surjection there exist elements z_1, z_2 such that $y_1 = \beta z_1$, $y_2 = \beta z_2$ and so

$$\sigma_A L_k^A \beta z_1 = \sigma_A L_k^A \beta z_2,$$

and by (1)

$$\sigma_B L_m^B \delta z_1 = \sigma_B L_m^B \delta z_2 .$$

Since $\sigma_B L_m^B$ is a bijection we have $\delta z_1 = \delta z_2$, and since α is a surjection there exists u such that $\alpha u = x$. Hence, by all preceding results it follows

$$\sigma_A L_{jk}^A (x, y_1) = \sigma_A L_{jk}^A \quad (x, y_2),$$

which means that $s\,\boldsymbol{i}\,t$ does not depend upon the choice of elements y_1 and y_2, and so the operation $\boldsymbol{i}$ is well defined.

In a similar way it can be shown that $\boldsymbol{i}$ is well defined in the case 4.

Since the corresponding mappings from 1—4 are bijections for every choice of elements $a_1, a_2, \ldots, a_{2n+1}$ it can easily be shown that $\boldsymbol{i}$ is a loop.

As in Lemma 1 from [7] we can prove that the loop $\boldsymbol{i}$ is a homotopic image of any other binary *GD*-groupoid L_{gh}^C from the class K_i and that the homotopy is of the form

$$\sigma_C L_{gh}^C (x, y) = \sigma_C L_g^C x \; \boldsymbol{i} \; \sigma_C L_h^C y.$$

Lemma 2. *If the eqivalence class K_i contains at least three derived binary GD-groupoids, such that three ternary GD-groupoids, from which these binary GD-groupoids are derived, are different, then the loop $\boldsymbol{i}$ is a group.*

The proof is analogous to the proof of Lemma 2 in [7].

Theorem. *Let $w_1 = w_2$ be a balanced law of the first kind in which all operations are ternary GD-groupoids. Let the set $\overline{\Phi}_0$ of all derived binary operations, which are not derived from the independent* ([7]) *ternary operations, be partitioned into equivalence classes by the relation $\sim : \overline{\Phi}_0 = K_1 \cup K_2 \ldots \cup K_p$. If in every class $K_i (i = 1, 2, \ldots, p)$, there exist two connected GD-groupoids L_{jk}^{Aqi} and L_{lm}^{Bri} with the property that among the following four pairs of mappings*

1) $\sigma_{A_{q_i}} L_j^{A_{q_i}}$, $\quad \sigma_{A_{q_i}} L_k^{A_{q_i}}$,

2) $\sigma_{B_{r_i}} L_l^{B_{r_i}}$, $\quad \sigma_{B_{r_i}} L_m^{B_{r_i}}$,

3) $\sigma_{A_{q_i}} L_j^{A_{q_i}}$, $\quad \sigma_{B_{r_i}} L_m^{B_{r_i}}$,

4) $\sigma_{A_{q_i}} L_k^{A_{q_i}}$, $\quad \sigma_{B_{r_i}} L_l^{B_{r_i}}$,

at least in one pair both mappings are bijections for every choice of elements $a_1, a_2, \ldots, a_{2n+1}$ *from the sets* $S_1, S_2, \ldots, S_{2n+1}$ *respectively, then there exists binary loop* $\boldsymbol{i}$, *defined on the set* S, *which is the homotopic image of all derived binary GD-groupoids from the class* K_i.

Then every ternary GD-groupoid A *from the set* $\Phi_1 \cup \Phi_2$, *except independent ternary GD-groupoids, can be represented in the form*

2) $$\sigma_A A(x, y, z) = \sigma_A L_1^A\ x\ \boldsymbol{i}\ (\sigma_A L_2^A\ y\ \boldsymbol{j}\ \sigma_A L_3^A\ z),$$

or

3) $$\sigma_A A(x, y, z) = (\sigma_A L_1^A\ x\ \boldsymbol{i}\ \sigma_A L_2^A\ y)\ \boldsymbol{j}\ \sigma_A L_3^A\ z.$$

The loops $\boldsymbol{i}$ *and* $\boldsymbol{j}$ *are uniquely determined,*

The expressions (2) *and* (3) *are equal then and only then when* $\boldsymbol{i} = \boldsymbol{j}$ *and the loop* $\boldsymbol{i}$ *is associative i.e. group.*

The loop $\boldsymbol{i}$ *is a group if the equivalence class* K_i *contains at least three derived binary GD-groupoids, such that three ternary GD-groupoids, from which these binary GD-groupoids are derived, are different.*

The proof is analogous to the proof of Theorem from [7].

This theorem can be applied to some balanced laws on n-ary quasigroups (by the method which is first used in [6]).

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