PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 20 (34), 1976, pp. 203—207

THE ADDITIVE EXHAUSTIVE FUNCTIONS ON M-LATTICE

Endre Pap

(Communicated June 27, 1975)

1. Introduction

P. Antosik gave in his paper [1] a uniform approach to Nikodym and Vitali-Hahn-Saks type theorems with additive set functions defined on a σ-ring of sets. We shall generalize some Antosik's results. Namely, we replace a σ-ring of sets by a M-lattice and then we shall define corresponding notions. We shall prove a theorem on uniform boundedness and two theorems on the pointwise convergence. As a basic tool in all proofs we shall use the Diagonal Theorem.

2. M-lattice

We shall define a particular lattice which generalize the σ-ring of sets. N always denotes the set of all positive integers.

Definition 1. *A partially ordered set M is a M-lattice if it has the following properties*:

(M_1) *M is a lattice (see G. Birkhoff* [2]), *i.e., to all $x, y \in M$ exist*

$$\sup(x, y) = x \vee y \quad \textit{and} \quad \inf(x, y) = x \wedge y \textit{ in } M;$$

$$(M_2) \qquad \{x_i\} \subset M \Rightarrow \sup_{i \in N} x_i \in M$$

(*we have also the notation* $\bigvee_{i=1}^{\infty} x_i$);

$$(M_3) \qquad \{y_i\} \subset M, \quad x \in M \Rightarrow x \wedge \sup_{i \in N} y_i = \sup_{i \in N} (x \wedge y_i)$$

(*the infinite distributive law*);

(M_4) *there exists $w \in M$ such that $w \leqslant x$ for every $x \in M$ (the existence of a least element*).

From (M_1) follows: If $x_i \in M$ for $i = 1, \ldots, n$ then $\bigwedge_{i=1}^{n} x_i \in M$ (G. Birkhoff [2]).

From (M_1), (M_2) and (M_3) follows

$$\sup_{n\in N} x_n \wedge \sup_{k\in N} y_k = \sup_{n,k\in N} (x_n \wedge y_k) \tag{1}$$

for $\{x_n\}\subset M$ and $\{y_k\}\subset M$ (B. Z. Vulih [3]).

We define some notions on M-lattice M corresponding to the notions on σ-ring of sets from Antosik's paper [1].

Definition 2. *Elements $x, y\in M$ are called disjoint iff hold $x\wedge y=w$.*

A sequence $\{x_n\}\subset M$ is called disjoint iff it is pairwise disjoint, i. e., $x_n\wedge x_m=w$ for each $n, m\in N$ and $n\neq m$.

S_0 is a collection of all disjoint sequences in M (C_0 on σ-ring in [1]).

We shall show that the collection S_0 satisfies the corresponding conditions to (i)—(iv) of Antosik's paper [1] for unconditional convergence.

Lemma. *A collection S_0 from M satisfies the following conditions:*

(I) *If $\{x_n\}\in S_0$, then, for each disjoint sequence $\{M_n\}\subset P(N)$ ($P(N)$ denotes the collection of all subsets of N)*

$$\{\bigvee_{j\in M_n} x_j\}\in S_0.$$

(II) *$\{x_n\}\in S_0$ implies that for each fixed $m\in N$, the sequence*

$$\{x_m\wedge \bigvee_{i=n+m}^{\infty} x_i\}_{n\in N}\in S_0.$$

(III) *For any sequence $\{y_n\}$ in M and $\{x_n\}\in S_0$ holds*

$$\{x_n\wedge y_n\}\in S_0.$$

(IV) *If $\{x_n\}\in S_0$, then $\{x_{m_n}\}\in S_0$.*

Proof. (I) By (1) we obtain for $n\neq m$

$$(\bigvee_{j\in M_n} x_j)\wedge(\bigvee_{k\in M_m} x_k)=\sup_{\substack{j\in M_n\\ k\in M_m}}(x_j\wedge x_k)=w,$$

because $\{x_n\}\in S_0$ and $M_n\cap M_m=\varnothing$ for $n\neq m$.

(II) For each fixed $m\in N$ and $\{x_n\}\in S_0$ we obtain by (M_2) for $n\neq s$

$$\big(x_m\wedge(\bigvee_{i=n+m}^{\infty} x_i)\big)\wedge\big(x_m\wedge(\bigvee_{r=s+m}^{\infty} x_r)\big)=$$

$$=\big(\bigvee_{i=n+m}^{\infty}(x_m\wedge x_i)\big)\wedge\big(\bigvee_{r=s+m}^{\infty}(x_m\wedge x_r)\big)=w.$$

(III) Let $\{x_n\}\in S_0$ and $\{y_n\}\subset M$, then by commutativity and associativity of $\wedge$ for $n\neq m$

$$(x_n\wedge y_n)\wedge(x_m\wedge y_m)=(x_n\wedge x_m)\wedge(y_n\wedge y_m)=w,$$

because $x_n\wedge x_m=w$ for $n\neq m$.

(IV) A trivial concequence of the definition 2.

3. Additive functions and Diagonal Theorem

Let X be a semi-topological semigroup with a zero element 0. This means that a semigroup X is endowed with topology such that the semigroup operation $(x, y) \to x+y$ is continuous in each variable separately.

A function $\mu: M \to X$ is said to be *exhaustive* (or S_0 — continuous) iff, for each sequence $\{x_n\} \in S_0$, the sequence $\mu(x_n)$ converges to 0 in X as $n \to \infty$. A function μ is *additive* iff, for each two disjoint elements $x, y \in M$, we have

$$\mu(x \vee y) = \mu(x) + \mu(y).$$

We denote the set of all additive and exhaustive functions on M-lattice M (with S_0) with values in X, with (MS_0, X).

If A and B are subsets of X, then by $A+B$ we understand the collections of all elements of the form $x+y$ with $x \in A$ and $y \in B$. If $A \subset X$ and $n \in N$, then we assume that $1A = A$ and $nA = (n-1)A + A$.

Diagonal Theorem. *Let $\mu_i \in (MS_0, X)$ $(i \in N)$, and let V_i $(i \in N)$ be open neighborhoods of the zero element in X and $\{x_i\} \in S_0$. If, for each $i \in N$, we have*

$$(\mu_i(x_i) + V_i) \cap V_i = \varnothing, \tag{2}$$

then there exists an infinite set $I \subset N$ and an element $x \in M$ such that

$$\mu_i(x) \notin V_i \tag{3}$$

for each $i \in I$.

In the proof of this Diagonal Theorem we shall use

Lemma (P. Antosik [1]). *Let $\mu_i \in (P(N) C_0, X)$, $(i \in N)$, and let V_i $(i \in N)$ be open neighborhoods of 0 in X. If, for each $i \in N$, we have*

$$(\mu_i(i) + V_i) \cap V_i = \varnothing$$

then there exists an infinite set $I \subset N$ and a set $J \subset I$ such that

$$\mu_i(J) \notin V_i$$

for each $i \in I$.

Proof of Diagonal Theorem. Let ν_i be functions such that, for each $i \in N$ and for each $K \in P(N)$, we have

$$\nu_i(K) = \mu_i(\bigvee_{j \in K} x_j) \quad (i \in N),$$

where $\{x_n\} \in S_0$. It is easy to prove that $\nu_i \in (P(N) C_0, X)$ (using our lemma).

By (2) we can write

$$(\nu_i(i) + V_i) \cap V_i = \varnothing.$$

Hence by Antosik's lemma, there exists an infinite set $I\subset N$ and a set $J\subset I$ such that

$$\nu_i(J)\notin V_i$$

for each $i\in I$, i.e.,

$$\mu_i(\bigvee_{j\in J} x_j)\notin V_i$$

for each $i\in I$. This completes the proof of the Diagonal Theorem.

4. Uniform boundedness on a particular set

G' denotes a semi-topological group, i.e., G' is a topological space such that the addition $(x, y)\to x+y$ is continuous in each variable separately and the mapping $x\to -x$ is continuous.

In this section we shall prove uniform boundedness theorem on particular sets of M, of pointwise bounded families of functions $\mu\in(MS_0, G')$.

Let F be a family of mappings from M into G' and let $A\subset M$. By $F[A]$ we understand the set of all points of the form $\mu(x)$ with $\mu\in F$ and $x\in A$. The family F is said to be *uniformly bounded* on A iff the set $F[A]$ is bounded (a subset $D\subset G'$ is said to be bounded iff for each neighborhood U of $0\in G'$ there exists a number $n\in N$ such that $D\subset nU$). The family F is said to be *pointwise bounded* on A iff for each element $x\in A$, the set $F[x]$ is bounded.

Now we can give the main result of this section.

Theorem 1. *Let F be a family of functions $\mu\in(MS_0, G')$. If the family F is pointwise bounded on M, then for each $\{x_n\}\in S_0$, it is uniformly bounded on the set*

$$A=\{x_1, x_2, \dots\}.$$

Proof. Suppose that the theorem is not true, i.e., there exists a sequence $\{x_n\}\in S_0$ such that the set $F[A]$ with $A=\{x_1, x_2, \dots\}$ is unbounded. Then there exists a symmetric neighborhood U of 0 (that is $U=-U$) such that for each $n\in N$ there exists an element $\mu_n\in F$ and an index i_n such that

$$\mu_n(x_{i_n})\notin 2nU.$$

Hence

$$(\mu_n(x_{i_n})+nU)\cap nU=\varnothing.$$

By (IV) from our lemma $\{x_{i_n}\}\in S_0$. Hence by Diagonal Theorem there exists an infinite set $I\subset N$ and an element $x\in G'$ such that

$$\mu_n(x)\notin nU$$

for each $n\in I$. A contradiction with a pointwise boundedness of the family F.

Remark 1. In particular if M is a σ-ring R, theorem 1 reduces to the Antosik's theorem [1] (the case: C_0) in which other important cases are included.

5. Vitali-Hahn-Saks Type Theorem

Let G be a topological abelian group.

In this section we shall prove that the set (MS_0, G) is a closed subgroup of the group of all additive functions (endowed with the usual addition of functions) under the pointwise convergence.

A sequence $\{g_n\} \subset G$ is Cauchy iff for each increasing sequence $\{p_n\} \subset N$, the sequence $\{g_{p_{n+1}} - g_{p_n}\}$ converges to 0 as $n \to \infty$.

Theorem 2. *Let* $\mu_i \in (MS_0, G)$ $(i \in N)$. *If for each* $x \in M$ *the sequence* $\{\mu_i(x)\}$ *is a Cauchy sequence then, for each sequence* $\{x_n\} \in S_0$, *the sequence* $\{\mu_i(x_j)\}$ *tends to* 0 as $j \to \infty$ *uniformly with respect to* i.

The proof of the theorem 2 is similar to the proof of Antosik's theorem 3 of [1] with some changes: sequences $\{E_n\}$ of sets we replace by sequences $\{x_n\} \in S_0$. Also, we replace 7. (1) with

$$\mu_i(x_i) \notin 3V,$$

and 7. (3) we replace with

$$\mu_{p_{j+1}}(x_{p_{j+1}}) - \mu_{p_j}(x_{p_{j+1}}) \notin 2V.$$

Hence by $\nu_i = \mu_{p_{i+1}} - \mu_{p_i}$ we obtain

$$(\nu_i(x_{p_{i+1}}) + V) \cap V = \varnothing.$$

Now, we make use of our Diagonal Theorem.

The main result of this section is.

Theorem 3. *Let* $\mu_i \in (MS_0, G)$ $(i \in N)$. *If, for each* $x \in M$,

$$\mu(x) = \lim_{i \to \infty} \mu_i(x),$$

then $\mu \in (MS_0, G)$.

The proof of the theorem 3 is similar to the proof of Antosik's theorem 4 of [1] with some changes: we replace sequences $\{E_n\}$ of sets by sequences $\{x_n\} \in S_0$ and theorem 3 of [1] by our theorem 2.

Remark 2. In particular if M is a σ-ring R, theorems 2 and 3 reduce respective to the Antosik's theorems 3 and 4 of [1] in the case C_0.

REFERENCES

[1] P. Antosik, *The Uniform Approach to Nikodym and Vitali-Hahn-Saks Type Theorems*, Studia math. (to appear).

[2] G. Birkhoff, *Lattice theory*, Coloquium Publications, V XXV, 1948.

[3] Б. З. Вулих, *Введение в теорию полуупорядоченных пространств*, Москва, 1961.

21 000 Novi Sad
Šekspirova 26
Yugoslavia