PUBLICATIONS DE L'INSTITUT MATHEMATIQUE
Nouvelle série, tome 20 (34), 1976, pp. 191—201

ON SOME FUNCTIONAL EQUATIONS AND THEIR APPLICATIONS

Prem Nath

(Received February 26, 1973)

1. Introduction

Let M denote the set of all *real-valued continuous* functions defined on the *closed interval* $I=[0, 1]$. T. W. Chaundy and J. B. Mcleod [3] proved that if a function $f\in M$ and satisfies the functional equation

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m} f(x_i)+\sum_{j=1}^{n} f(y_j),\quad x_i\geqslant 0,\quad y_j\geqslant 0,\quad \sum_{i=1}^{m} x_i=\sum_{j=1}^{n} y_j=1;$$
$$m, n=1, 2, 3, \ldots$$

then f is of the form

$$f(x)=cx\log_b x \qquad x\in(0, 1],\ b>1,\ c \text{ an arbitrary constant},$$
$$=0, \qquad x=0.$$

Our main object in this paper (§ 2, § 3) is to discuss the continuous solutions of some functional equations which are generalizations of the above functional equation. In § 2, we have proved a lemma which seems to be quite trivial but we have made use of it extensively. This lemma enables us to find continuous solutions of many other functional equations discussed in § 4.

Throughout the paper, we shall use the fact that $0^a=0$ whenever $a>0$.

Some Functional Equations useful in Information theory.

In this section, we shall find continuous solutions of some functional equations connected with Shannon's entropy and its generalizations. We prove the following theorem:

Theorem 1. *If, for all positive integers m and n, the functions $f\in M$, $g\in M$ and $h\in M$ satisfy the functional equations*

$$(A_1)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m} f(x_i)+\left(\sum_{i=1}^{m} x_i^{\alpha}\right)\left(\sum_{j=1}^{n} f(y_j)\right)+d\sum_{i=1}^{m} x_i^{\alpha},\qquad \alpha>0,$$

$$(A_2)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m} f(x_i)+\left(\sum_{i=1}^{m} x_i^{\alpha}\right)\left(\sum_{j=1}^{n} g(y_j)\right),\qquad \alpha>0,$$

$$(A_3)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m} f(x_i)+\left(\sum_{i=1}^{m} x_i^{\alpha}\right)\left(\sum_{j=1}^{n} h(y_j)\right),\qquad \alpha>0,$$

where $x_i\geqq 0$, $y_j\geqq 0$, $\sum_{i=1}^{m} x_i=\sum_{j=1}^{n} y_j=1$ *and* $d\in R=(-\infty, +\infty)$, *then*

$$(2.1)\qquad \begin{cases} f(x)=cx\log_b x-dx, & x\in(0,1),\ \alpha=1,\ b>1,\\ \quad =c(x^{\alpha}-x)-dx, & x\in(0,1),\ \alpha\neq 1,\ \alpha>0,\\ \quad =0, & x=0,1;\ \ \alpha>0,\end{cases}$$

$$(2.2)\qquad \begin{cases} f(x)=cx\log_b x-dx, & g(x)=cx\log_b x; & x\in(0,1),\ \alpha=1,\\ \quad =c(x^{\alpha}-x)-dx, & \quad =c(x^{\alpha}-x); & x\in(0,1),\ \alpha>0,\ \alpha\neq 1,\\ \quad =0, & \quad =0, & x=0,\ \alpha>0,\\ \quad =-d & \quad =0 & x=1,\ \alpha>0,\end{cases}$$

$$(2.3)\qquad \begin{cases} f(x)=cx\log_b x-dx, & g(x)=cx\log_b x-c_1x, & h(x)=cx\log_b x-c_2x,\\ & & x\in(0,1),\ \alpha=1,\\ \quad =c(x^{\alpha}-x)-dx, & \quad =c(x^{\alpha}-x)+(c_2-d)x, & \quad =c(x^{\alpha}-x)-c_2x,\\ & & x\in(0,1),\ \alpha>0,\ \alpha\neq 1,\\ \quad =0, & \quad =0, & \quad =0,\quad x=0,1;\ \ \alpha>0,\end{cases}$$

are respectively the continuous solutions of (A_1), (A_2) *and* (A_3) *with*

$$(2.4)\qquad d=-f(1),\quad c_1=-g(1),\quad c_2=-h(1),\quad c \text{ an arbitrary constant.}$$

To prove the above theorem, we need the following:

Lemma. *Let* $F\in M$, $G\in M$. *The necessary and sufficient condition that*

$$(2.5)\qquad \sum_{i=1}^{m} F(x_i)=\sum_{i=1}^{m} G(x_i),\quad x_i\geqq 0,\quad \sum_{i=1}^{m} x_i=1,\qquad m=1,2,3,\dots$$

is that

$$F(x)=G(x)\quad \textit{for all}\quad x\in I=[0,1].$$

Proof. The sufficiency part is obvious. To prove that the condition is necessary, we proceed as follows. Let $U: I\to R$ such that

$$(2.6)\qquad U(x)=F(x)-G(x),\quad x\in I.$$

Clearly, $U\in M$ and (2.5) reduces to

$$(2.7)\qquad \sum_{i=1}^{m} U(x_i)=0,\quad x_i\geqq 0,\quad \sum_{i=1}^{m} x_i=1,\qquad m=1,2,3,\dots$$

We shall prove that

$$U(x)=0 \quad \text{for all} \quad x\in I. \tag{2.8}$$

Let $m=1$. Clearly, $x_1=1$ and (2.7) gives

$$U(1)=0. \tag{2.9}$$

Now, let $m=2$, $x_1=1$, $x_2=0$. Then, (2.7) and (2.9) give

$$U(0)=0. \tag{2.10}$$

Let $y\in(0, 1)$ be a rational number such that $y=\frac{r}{u}$, r and u being positive integers. Putting

$$m=u-r+1, \quad x_1=y=\frac{r}{u}, \quad x_2=x_3=\cdots=x_{u-r+1}=\frac{1}{u}, \quad 1\leqslant r<u,$$

(2.7) reduced to

$$U\left(\frac{r}{u}\right)+(u-r)\,U\left(\frac{1}{u}\right)=0. \tag{2.11}$$

If we put $r=1$, then (2.11) gives

$$U\left(\frac{1}{u}\right)=0. \tag{2.12}$$

From (2.11) and (2.12), it follows that

$$U\left(\frac{r}{u}\right)=0. \tag{2.13}$$

Since $U\in M$, therefore, by the continuity of U, (2.13) gives

$$U(x)=0 \quad \text{for all real numbers} \quad x\in(0, 1). \tag{2.14}$$

Equations (2.9), (2.10) and (2.14), taken together, are equivalent to (2.8). The conclusion follows from (2.6) and (2.8).

Now we give the proof of the theorem.

For (A_1), let us define

$$F(x)=\sum_{j=1}^{n} f(xy_j), \quad G(x)=f(x)+x^{\alpha}\sum_{j=1}^{n} f(y_j)+dx^{\alpha}, \quad \alpha>0, \quad x\in I.$$

Clearly, (A_1) reduces to (2.5). Since $f\in M \Rightarrow F\in M$, $G\in M$, therefore, by the lemma,

$$\sum_{j=1}^{n} f(xy_j)=f(x)+x^{\alpha}\sum_{j=1}^{n} f(y_j)+dx^{\alpha}, \quad \alpha>0, \; x\in I, \; d\in R, \; y_j\geqslant 0, \; \sum_{j=1}^{n} y_j=1. \tag{2.15}$$

Putting $x=1$, (2.15) gives

$$f(1)=-d. \tag{2.16}$$

Also, if $n=2$, $x=0$, $y_1=1$, $y_2=0$, then (2.15) gives

$$f(0)=0. \tag{2.17}$$

Let $n=v-s+1$, $1\leqslant s<v$, v and s being positive integers, and $y=\frac{s}{v}$ be a rational number lying in (0, 1). Choosing

$$y_1=y=\frac{s}{v}, \quad y_2=y_3=\cdots=y_{v-s+1}=\frac{1}{v},$$

the functional equation (2.15) reduces to

$$f\left(x\cdot\frac{s}{v}\right)+(v-s)f\left(x\cdot\frac{1}{v}\right)=f(x)+x^{\alpha}\left[f\left(\frac{s}{v}\right)+(v-s)f\left(\frac{1}{v}\right)\right]+dx^{\alpha}, \tag{2.18}$$
$$x\in I, \quad \alpha>0, \quad d\in R.$$

Putting $s=1$, (2.18) gives

$$f\left(x\cdot\frac{1}{v}\right)=\frac{1}{v}f(x)+x^{\alpha}f\left(\frac{1}{v}\right)+dx^{\alpha}\cdot\frac{1}{v}, \quad \alpha>0, \quad x\in I, \quad d\in R. \tag{2.19}$$

From (2.18) and (2.19), it follows that

$$f\left(x\cdot\frac{s}{v}\right)=\frac{s}{v}f(x)+x^{\alpha}f\left(\frac{s}{v}\right)+dx^{\alpha}\frac{s}{v}, \quad x\in I, \quad \alpha>0, \quad d\in R. \tag{2.20}$$

Hence, by continuity of f,

$$f(xy)=yf(x)+x^{\alpha}f(y)+dx^{\alpha}y, \quad x\in I, \quad y\in(0, 1), \quad \alpha>0, \quad d\in R. \tag{2.21}$$

In particular,

$$f(xy)=yf(x)+x^{\alpha}f(y)+dx^{\alpha}y, \quad x, y\in(0, 1), \quad \alpha>0, \quad d\in R. \tag{2.22}$$

Case 1. Let $\alpha=1$. Then, (2.22) reduces to

$$f(xy)=yf(x)+xf(y)+dxy, \quad x, y\in(0, 1), \quad \alpha>0, \quad d\in R,$$

whose *continuous solutions* are of the form

$$f(x)=cx\log_b x-dx, \quad x\in(0, 1), \quad b>1, \quad d\in R, \tag{2.23}$$

where c is an arbitrary real constant.

Case 2. Let $\alpha\neq 1$. Then, (2.22) gives

$$yf(x)+x^{\alpha}f(y)+dx^{\alpha}y=xf(y)+y^{\alpha}f(x)+dy^{\alpha}x, \quad x, y\in(0, 1), \quad \alpha>0, \tag{2.24}$$
$$\alpha\neq 1, \quad d\in R.$$

If we put

$$h_1(x)=\frac{f(x)}{x}+d, \quad x\in(0, 1), \quad d\in R. \tag{2.25}$$

then (2.24) reduces to

$$h_1(x)+x^{\alpha-1}h_1(y)=h_1(y)+y^{\alpha-1}h_1(x), \quad x, y\in(0, 1), \quad \alpha>0, \quad \alpha\neq 1,$$

from which it follows that

$$h_1(x)=c(x^{\alpha-1}-1), \quad x\in(0, 1), \quad \alpha>0, \quad \alpha\neq 1, \quad c \text{ an arbitrary constant.}$$

Consequently

$$f(x)=c(x^{\alpha}-x)-dx, \quad x\in(0, 1), \quad \alpha>0, \; \alpha\neq 1, \; c \text{ an arbitrary constant} \tag{2.26}$$

Equations (2.16), (2.17), (2.23) and (2.26), taken together, constitute the required solution (2.1) of (A_1).

Out of (A_2) and (A_3) we shall give the proof for (A_3) only. The proof for (A_2) follows on similar lines. Let us define

$$F(x)=\sum_{j=1}^{n} f(xy_j), \quad G(x)=g(x)+x^{\alpha}\sum_{j=1}^{n} h(y_j), \quad x\in I, \;\alpha>0, \; y_j\geqslant 0, \; \sum_{j=1}^{n} y_j=1.$$

Clearly, (A_3) reduces to (2.5). Also, $f\in M$, $g\in M$, $h\in M \Rightarrow F\in M$, $G\in M$. Hence, by the lemma,

$$\sum_{j=1}^{n} f(xy_j)=g(x)+x^{\alpha}\sum_{j=1}^{n} h(y_j), \quad x\in I, \quad \alpha>0, \quad y_j\geqslant 0, \quad \sum_{j=1}^{n} y_j=1. \tag{2.27}$$

Putting $x=1$, (2.27) gives

$$f(x)=g(1)+h(x), \qquad x\in I. \tag{2.28}$$

Similarly

$$f(x)=h(1)\,x^{\alpha}+g(x), \qquad x\in I, \quad \alpha>0. \tag{2.29}$$

Elimination of g and h from (A_3), (2.28) and (2.29) gives rise to (A_1) with

$$d=c_1+c_2, \quad c_1=-g(1), \quad c_2=-h(1), \quad f(1)=g(1)+h(1).$$

Hence, f is given by (2.1). The forms of g and h can be easily found from (2.1), (2.28) and (2.29).

This completes the proof of the theorem.

3. Applications to Information theory

In this section, we shall point out the usefulness of the functional equations (A_1), (A_2) and (A_3) in information theory.

If $\alpha=1$ and $d=0$, then (A_1) reduced to

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m} f(x_i)+\sum_{j=1}^{n} f(y_j), \quad x_i\geqslant 0, \quad y_j\geqslant 0, \quad \sum_{i=1}^{m} x_i=\sum_{j=1}^{n} y_j=1, \quad m, n=1, 2, 3, \ldots \tag{A_4}$$

and from (2.1), it follows that the continuous solutions of (A_4) are of the form

$$\begin{aligned} f(x)&=cx\log_b x, \qquad b>1, \quad x\in(0, 1], \quad c \text{ an arbitrary constant.} \\ &=0 \qquad \text{if} \quad x=0. \end{aligned} \tag{3.1}$$

The functional equation (A_4) is due to T. W. Chaundy and J. B. Mcleod [3] who came it across while making their studies in statistical thermodynamics. They proved that if $f \in M$ satisfies (A_4) for all positive integers m and n, then f is of the form (3.1). We have also proved the same result but our technique is different. With $f \in M$, the functional equation (A_4) has also been dealt with by (i) J. Aczél and Z. Daróczy [2] for $n=m$, (ii) Daróczy [6] for $m=2$, $n=1, 2, \ldots$ (iii) Kannappan [8] for all positive integers $m \geq 1$, $n \geq 1$. Recently, Daróczy [5] has weakened the requirement of continuity and assumed f to be measurable in (0, 1) and $n=m=1$ and $m=2$, $n=3$.

If $f \in M$ in (A_4) satisfies the additional condition

$$f\left(\frac{1}{2}\right)=\frac{1}{2}, \tag{3.2}$$

then (3.1) becomes (with base $b=2$)

$$\begin{aligned} f(x) &= -x \log_2 x, & x \in (0, 1], \\ &= 0, & x = 0, \end{aligned} \tag{3.3}$$

and the quantity $\sum_{i=1}^{m} f(x_i) = -\sum_{i=1}^{m} x_i \log_2 x_i$ represents Shannon's entropy $H_1(\mathfrak{X})$ of the complete probability distribution $\mathfrak{X}=(x_1, x_2, \ldots, x_m)$, $x_i \geq 0$, $\sum_{i=1}^{m} x_i = 1$.

If $d=0$, $\alpha>0$, $\alpha \neq 1$, then (A_1) reduces to

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{m} f(x_i) + \left(\sum_{i=1}^{m} x_i^{\alpha}\right)\left(\sum_{j=1}^{n} f(y_j)\right), \tag{A_5}$$

$$x_i \geq 0, \quad y_j \geq 0, \quad \sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1.$$

From (2.1), it follows that the continuous solutions of (A_5), satisfying (3.2), are of the form $f=z_\alpha$ where

$$\begin{aligned} z_\alpha(x) &= \frac{x-x^\alpha}{1-2^{1-\alpha}}, \quad \alpha>0, \quad \alpha \neq 1, \quad x \in (0, 1), \\ &= 0, \qquad x=0, 1; \qquad \alpha>0, \quad \alpha \neq 1. \end{aligned} \tag{3.4}$$

Clearly,

$$\sum_{i=1}^{m} z_\alpha(x_i) = \frac{1-\sum_{i=1}^{m} x_i^\alpha}{1-2^{1-\alpha}}, \quad \alpha>0, \quad \alpha \neq 1, \quad x_i \geq 0, \quad \sum_{i=1}^{m} x_i = 1. \tag{3.5}$$

The RHS in (3.5) is the non-additive entropy $I_\alpha(\mathfrak{X})$, $\alpha>0$, $\alpha \neq 1$, due to Havrda and Charvat [1] and Z. Daróczy [4]. Thus, when $d=0$, the resulting form of (A_1) enables us to characterize simultaneously the entropies $H_1(\mathfrak{X})$, and $I_\alpha(\mathfrak{X})$, $\alpha>0$, $\alpha \neq 1$. Daróczy [4] characterized $I_\alpha(\mathfrak{X})$ by using a generalized form of fundamental equation of information theory.

In general, if f, g and h are non-identically vanishing continuous functions satisfying (A_1), (A_2) and (A_3), then the sums of the forms $\sum_{i=1}^{m} f(x_i)$, $\sum_{i=1}^{m} g(x_i)$, $\sum_{i=1}^{m} h(x_i)$ represent a quantity of the form $H_1(\mathfrak{X})+k$, $k\in R$, when $\alpha=1$, and a quantity of the form $I_\alpha(\mathfrak{X})+k$, $k\in R$, $\alpha>0$, $\alpha\neq 1$, where $H_1(\mathfrak{X})$ and $I_\alpha(\mathfrak{X})$ now denote Shannon's entropy (upto a multiplicative constant $c\neq 0$ and with base $b>1$ and non-additive entropy due to Havrda and Charvat (up to a multiplicative constant $c\neq 0$). In other words, $H_1(\mathfrak{X})$ and $I_\alpha(\mathfrak{X})$ denote the quantities $c\sum_{i=1}^{m} x_i \log_b x_i$ and $c\left(\sum_{i=1}^{m} x_i^\alpha - 1\right)$, $\alpha>0$, $\alpha\neq 1$, $c\neq 0$, respectively.

The functional equation (A_1) also occurs in the solutions of (A_2) and (A_3) which contain two and three unknown functions respectively. For this reason, it is an important generalization of (A_4).

Now we state the following theorem:

Theorem 2. *If, for all positive integers $m=1, 2, \ldots$ the functions $f\in M$, $g\in M$, satisfy the functional equations*

$$(A_6)\qquad \sum_{i=1}^{m} f(x_i)=\lambda \sum_{i=1}^{m} x_i \log_b x_i,\quad b>1,\quad x_i\geqq 0,\quad \sum_{i=1}^{m} x_i=1,\quad \lambda\in R,$$

$$(A_7)\qquad \sum_{i=1}^{m} g(x_i)=\lambda\left(\sum_{i=1}^{m} x_i^\alpha - 1\right),\quad \alpha>0,\quad \alpha\neq 0,\quad \lambda\in R,\quad x_i\geqq 0,\quad \sum_{i=1}^{m} x_i\geqq 1,$$

then

$$(3.6)\qquad f(x)=\lambda x \log_b x,\quad b>1,\quad x\in I,\quad \lambda\in R,$$

$$(3.7)\qquad g(x)=\lambda(x^\alpha - x),\quad \alpha>0,\quad \alpha\neq 1,\quad x\in I,\quad \lambda\in R.$$

The proof of this theorem is omitted as the required conclusions follow by the direct application of the lemma.

4. Some functional equations whose continuous solutions are straight lines passing through the origin.

In this section, we give some functional equations whose continuous solutions are straight lines passing through the origin.

According to theorem 2 on page 48 in [1], it is known that the continuous solutions of Cauchy's equation

$$(4.1)\qquad f(x_1+x_2+\cdots+x_n)=\sum_{j=1}^{n} f(x_j)$$

on any arbitrary interval $[\alpha, \beta]$ are

$$(4.2)\qquad f(x)=cx,\quad c \text{ an arbitrary constant},\quad x\in[\alpha, \beta],$$

if $[\alpha, \beta]\cap[n\alpha, n\beta]\neq\varnothing$. If $\alpha=0$, $\beta=1$, then $[0, 1]\cap[0, n]\neq\varnothing$ for all positive integers $n\geqq 1$ so that $f\in M$. We discuss below some more functional equations whose continuous solutions are linear.

We prove the following theorem:

Theorem 3. *If $f \in M$ and is defined as*

$$f(x) = cx, \quad x \in I, \quad c \text{ an arbitrary constant}, \tag{4.3}$$

then, for all positive integers m and n, f satisfies the functional equations

$$\sum_{i=1}^{m} f(x_i) = c, \tag{B_1}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = c, \tag{B_2}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{m} f(x_i), \tag{B_3}$$

$$\sum_{i=1}^{m} \sum_{i=1}^{n} f(x_i y_j) = \sum_{j=1}^{n} f(y_i), \tag{B_4}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \frac{1}{2}\left[\sum_{i=1}^{m} f(x_i) + \sum_{j=1}^{n} f(y_j)\right], \tag{B_5}$$

where $x_i \geqslant 0$, $y_j \geqslant 0$, $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1$. *Conversely, the only continuous solutions of* (B_1) *to* (B_5) *are of the form* (4.3).

Proof. The fact that f, given by (4.3), satisfies (B_1) to (B_5) is a matter of simple verification. Hence, we need to prove only the converse part.

As regards (B_1), all that we need is to define $F(x) = f(x)$, $G(x) = cx$, $x \in I$. Then, (B_1) reduces to (2.5) and the conclusion follows by the application of the lemma.

For (B_2), we need to define

$$F(x) = \sum_{j=1}^{n} f(xy_j), \quad G(x) = c \sum_{j=1}^{n} xy_j, \quad x \in I, \quad y_j \geqslant 0, \quad \sum_{j=1}^{n} y_j = 1.$$

Then, (B_2) reduces to (2.5). Since, the conditions of the lemma are satisfied, therefore

$$\sum_{j=1}^{n} f(xy_j) = \sum_{j=1}^{n} cxy_j, \quad x \in I, \quad y_j \geqslant 0, \quad \sum_{j=1}^{n} y_j = 1. \tag{4.4}$$

In particular, if $x = 1$, (4.4) reduces to

$$\sum_{j=1}^{n} f(y_j) = \sum_{j=1}^{n} cy_j, \quad y_j \geqslant 0, \quad \sum_{j=1}^{n} y_j = 1, \qquad n = 1, 2, \ldots .$$

Using the lemma again, we get (4.3).

For (B_3), we need to define

$$F(x) = \sum_{j=1}^{n} f(xy_j), \quad G(x) = f(x), \quad x \in I, \quad y_j \geqslant 0, \quad \sum_{j=1}^{n} y_j = 1.$$

Following the above arguments, it follows that

$$\sum_{j=1}^{n} f(xy_j)=f(x), \quad x\in I, \ y_j\geqq 0, \ \sum_{j=1}^{n} y_j=1. \tag{4.5}$$

In particular, if we put $x=1$, $f(1)=c$, we get $\sum_{j=1}^{n} f(y_j)=c$. Consequently, (4.3) follows.

We omit the proof of (B_4). For (B_5), we define

$$F(x)=\sum_{j=1}^{n} f(xy_j), \quad G(x)=\frac{1}{2}\left[f(x)+x\sum_{j=1}^{n} f(y_j)\right], \quad x\in I, \ y_j\geqq 0, \ \sum_{j=1}^{n} y_j=1.$$

By the usual arguments

$$\sum_{j=1}^{n} f(xy_j)=\frac{1}{2}\left[f(x)+x\sum_{j=1}^{n} f(y_j)\right], \quad x\in I, \ y_j\geqq 0, \ \sum_{j=1}^{n} y_j=1. \tag{4.6}$$

Putting $x=1$, $f(1)=c$, (4.6) gives

$$\sum_{j=1}^{n} f(y_j)=\sum_{j=1}^{n} \hat{f}(y_j), \quad y_i\geqq 0, \ \sum_{j=1}^{n} y_j=1, \tag{4.7}$$

where

$$\hat{f}(x)=\frac{1}{2}cx+\frac{1}{2}f(x).$$

Since $f\in M$, $\hat{f}\in M$, therefore the application of the lemma to (4.7) gives

$$f(x)=\frac{1}{2}cx+\frac{1}{2}f(x), \quad x\in I,$$

from which (4.3) follows immediately. This completes the proof of the theorem.

In functional equations (B_1) to (B_5), there is only one unknown function. The next theorem deals with some more functional equations involving more than one unknown continuous functions and yet having continuous solutions as straight lines passing through the origin.

Theorem 4. *If, for all positive integers m and n, the functions f, g, h, all belonging to M, satisfy the functional equations*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m} g(x_i) \tag{B_6}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{j=1}^{n} h(y_j) \tag{B_7}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\frac{1}{2}\left[\sum_{i=1}^{m} f(x_i)+\sum_{j=1}^{n} g(y_j)\right], \tag{B_8}$$

where $x_i \geqslant 0$, $y_j \geqslant 0$, $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1$, *then*

$$f(x) = g(x) = cx, \qquad x \in I, \tag{4.8}$$

$$f(x) = h(x) = cx, \qquad x \in I, \tag{4.9}$$

$$f(x) = g(x) = cx, \qquad x \in I, \tag{4.10}$$

are respectively the continuous solutions of (B_6), (B_7) *and* (B_8) *respectively.*

The proof for (B_6) and (B_7) is omitted because it is simple. For (B_8), we define

$$F(x) = \sum_{j=1}^{n} f(xy_j), \quad G(x) = \frac{1}{2}\left[f(x) + x\sum_{j=1}^{n} g(y_j)\right], \quad x \in I, \ y_j \geqslant 0, \ \sum_{j=1}^{n} y_j = 1.$$

Now, $f \in M$, $g \in M \Rightarrow F \in M$, $G \in M$. Also, (B_8) reduces to (2.5). Therefore, the lemma gives

$$\sum_{j=1}^{n} f(xy_j) = \frac{1}{2}\left[f(x) + x\sum_{j=1}^{n} g(y_j)\right], \quad x \in I, \ y_j \geqslant 0, \ \sum_{j=1}^{n} y_j = 1. \tag{4.11}$$

Putting $x = 1$, $f(1) = c$, (4.11) gives

$$\sum_{j=1}^{n} f(y_j) = \sum_{j=1}^{n} \hat{g}(y_j), \quad y_j \geqslant 0, \ \sum_{j=1}^{n} y_j = 1, \tag{4.12}$$

where

$$\hat{g}(x) = \frac{1}{2} cx + \frac{1}{2} g(x), \qquad x \in I.$$

Since $f \in M$, $\hat{g} \in M$, therefore, by the lemma, (4.12) gives

$$f(x) = \frac{1}{2} cx + \frac{1}{2} g(x). \tag{4.13}$$

But, when $n = 1$, (4.11) gives $f(x) = xg(1)$ so that $g(1) = f(1) = c$. Thus, $f(x) = cx$, $x \in I$. Substitution in (4.13) gives $g(x) = f(x)$, $x \in I$. Thus, (4.10) is proved.

The functional equations discussed in theorems 1 to 4 can be solved by the other alternative methods also. We hope to present them elsewhere.

Addendum. In their joint works, Behara & Nath [9], [10] have also given the following two generalizations

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j) = \left(\sum_{j=1}^{n} y_j^{\alpha}\right)\left(\sum_{i=1}^{m} f(x_i)\right) + \left(\sum_{i=1}^{m} x_i^{\beta}\right)\left(\sum_{j=1}^{n} f(y_j)\right) \tag{A_8}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{m} f(x_i) + \sum_{j=1}^{n} f(y_j) + (2^{1-\alpha} - 1)\left(\sum_{i=1}^{m} f(x_i)\right)\left(\sum_{j=1}^{n} f(y_j)\right) \tag{A_9}$$

where $\alpha > 0$, $\beta > 0$, $x_i \geqslant 0$, $y_i \geqslant 0$, $\sum_{i=1}^{m} x_i = \sum_{j=1}^{m} y_i = 1$. Both ($A_8$) and ($A_9$) reduce to ($A_4$) when $\alpha = \beta = 1$ respectively. Also, (A_5) is a special case of (A_8).

REFERENCES

[1] J. Aczél, *Lectures on functionel equations and their applications*, Acad. Press, New York, 1976.

[2] J. Aczél and Z. Daróczy *Characterisierung der Entropien positiver Ordnung und der Shannonschen Entropie*, Acta Math. Sci. Hung., 14, 95—121, 1963.

[3] T. W. Chaundy and J. B. Mcleod, *On a functional equation*, Proc. Edin. Math. Notes, 43, 7—8, 1960.

[4] Z. Daróczy, *Generalized information functions*, Information and Control, 16, 36—51, 1970.

[5] Z. Daróczy, *On the measurable solutions of a functional equation*, Acta Math. Sci. Hung., 22, 11—14, 1971.

[6] Z. Daróczy, *Über die characterisierung der Shannonschen Entropie*, Proc. of the Colloquium on Information theory, Debrecen, János Bolyai Math. Soc. Budapest, 1, 135—139, 1968.

[7] J. Havrda and F. Charvat, *Quantification method of classification processes, the concept of structural a-entropy*, Kybernetika, 3, 30—35, 1967.

[8] PL. Kannappan, *On Shannon's entropy, directed divergence and inaccuracy*, Zeit. für Wahr. und Verw. Geb., 22, 95—100, 1972.

[9] P. Nath and M. Behara, *Additive and Non-additive entropies of finite measurable partitions*, Prob. and Information Theory, Vol II, 102—138, Lecture Notes in Mathematics, 296, Springer-Verlag, 1973.

[10] P. Nath and M. Behara, *Information and Entropy of countable measurable partitions I*, Kybernetika, 10, 491—503, 1974.

[11] C. E. Shannon, *A mathematical theory of communication*, Bell Syst. Tech. Jour. 27, 378—423, 623—656, 1948.

PREM NATH,
Faculty of Mathematics,
Delhi University, Delhi-7,
INDIA.