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# FIXED POINTS OF LOCAL CONTRACTIONS

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**1. Introduction.** Let $M$ be a metrizable topological space and $D=D(M)$ the family of all topologically equivalent metrics on $M$. For $\varepsilon>0$ and $d\in D$, $M$ is said to be *$(d, \varepsilon)$-chainable* (or connected in the Cantor sense) if for each two $x$ and $y$ in $M$ there is a finite sequence of elements in $M$ $\zeta=\{z_1, z_2, \ldots, z_n\}$; $z_0=x$, $z_n=y$ and

$$d(z_i, z_{i+1})\leq\varepsilon, \quad i=0, 1, \ldots, n-1.$$

In [1], M. Edelstein uses this concept to give an extension of Banach's contraction principle.

For $U\subseteq M\times M$ (and it can be supposed that $U$ contains the diagonal of $M\times M$ and $U=U^{-1}$) $M$ is *$U$-chainable* if for each two $x$ and $y$ in $M$, there is a finite sequence of elements in $M\zeta=\{z_1, z_2, \ldots, z_n\}$; $z_0=x$ and $z_n=y$ such that $(z_i, z_{i+1})\in U$, $i=0, 1, \ldots, n-1$. The sequence $\zeta$ will be called a $U$-chain from $x$ to $y$. This concept does not depend on the metric $d\in D$ and so is not metrical but combinatorial in the sense that if $M$ is $U$-chainable and $g:M\rightarrow M'$ is one-to-one and onto, then $M'$ is $g(U)$-chainable, with $g(x, y)=(gx, gy)$. It follows at once that for $\varepsilon>0$ and $d\in D$ and $U=\{(x, y);\ d(x, y)<\varepsilon\}$, $M$ is $U$-chainable if and only if $M$ is $(\varepsilon, d)$-chainable. In [3], a further extension of the mentioned result of Edelstein has ben given for $U$-chainable metric spaces, with the concept of *$U$-contraction* to mean a mapping $f:(M, d)\rightarrow(M, d)$ such that

$$d(fx, fy)\leq q\, d(x, y), \quad 0\leq q<1 \text{ and } (x, y)\in U$$

and satisfying an extra condition that $fU\subseteq U$. The aim of this note is to eliminate this extra condition for a large class of metric spaces and to prove a contraction type theorem under weakened metric and strenghtened topological conditions.

**2. Definitions and formulation of the theorem.** A space $M$ is *arcwise connected* if for each $x$ and $y$ in $M$, there is a homeomorphism $h:[0, 1]\rightarrow M$ such that $h(0)=x$ and $h(1)=y$. Call $L=h([0, 1])$ an arc from $x$ to $y$.

A *subdvision* $S$ of $[0, 1]$ is a finite family of closed intervals covering $[0, 1]$, such that any two intervals have at most one point in common. For $d\in D$ fixed, the length of $L$ is the number

$$\|L\|=\sup_S\{\sum d(h(t_i), h(t_{i+1})) \quad i=0, \ldots, n-1\},$$

where $t_n = 1$. Being $L$ fixed, it is easy to see that $\|L\|$ is independent of the special choice of homeomorphism $h$. When $\|L\| < +\infty$, $L$ is an *arc of finite length* and a metric space $(M, d)$ is *connected by finite arcs* if for each $x$ and $y$ in $M$ there is an $L$ from $x$ to $y$ such that $\|L\| < +\infty$.

A mapping $f:(M, d) \to (M, d)$ is a *local contraction* if there is a $q$, $0 \leq q < 1$ such that the following condition is satisfied:

(LC) *For each* $x \in M$, *there exists* $S(x, \varepsilon(x)) = \{y : d(x, y) < \varepsilon(x)\}$ such that

$$d(fu, fv) \leq q\, d(u, v) \quad \textit{for} \quad u, v \in S(x, \varepsilon(x));$$

and $f$ is a *strong lacal contraction* if the condition:

(SLC) For each $x \in M$, there exists $S(x, \varepsilon(x))$ such that

$$d(f^n u, f^n v) \leq q^n K(u, v), \quad \text{for} \quad u, v \in S(x, \varepsilon(x)),$$

*where* $n = 1, 2, \ldots$ *and* $K(u, v)$ *is a number depending on* $u$ *and* $v$, is satisfied.

Note that when $U$ is a neighborhood of diagonal in $M \times M$ and $f$ is a $U$-contraction and $fU \subseteq U$, then the condition (SLC) is satisfied with $K(x, y) = d(x, y)$. The following proposition shows how the above conditions are related to each other.

Proposition. *Let* $f:(M, d) \to (M, d)$ *be a local contraction. If all iterates of* $f$ *converge to the same point* $x_0 \in M$, *then* $f$ *is a strong local contraction.*

Proof. Let $\varepsilon$ be such that

$$d(fu, fv) \leq q d(u, v) \quad \text{for} \quad u, v \in S(x_0, \varepsilon).$$

For any two $x, y$ in $M$; there is an $m$ such that $f^m x, f^m y \in S(x_0, \varepsilon)$. Since $f x_0 = x_0$, we have

$$d(f^{m+1} x, x_0) = d(f(f^m x), f x_0) \leq q\, \varepsilon,$$

and that $f^n x \in S(x_0, \varepsilon)$, $f^n y \in S(x_0, \varepsilon)$ for $n > m$, is proved by induction. Now for $n > m$,

$$d(f^n x, f^n y) \leq q^{n-m} d(f^m x, f^m y) = q^n \frac{d(f^m x, f^m y)}{q^m}.$$

Put

$$K(x, y) = \frac{\max\{d(f^i x, f^i y) : i \leq m\}}{q^m}.$$

and we will have $d(f^n x, f^n y) \leq q^n K(x, y)$ for each $n$.

**3.** Now we can formulate this theorem.

Theorem. *Let* $M$ *be a metrizable, arcwise connected topological space and* $f: M \to M$. *If there is a* $d \in D(M)$ *such that* $(M, d)$ *is complete and connected by finite arcs and* $f$ *is a local contraction, then all iterates converge to a unique fixed point of* $f$.

*Proof of the theorem.* The proof will follow from three lemmas, the second of which is of some independent interest.

Lemma 1. *A topological space $M$ is connected iff $M$ is $U$-chainable for each $U$ which is a neighborhood of the diagonal in $M \times M$* (well known).

Proof. Let $M$ be connected and $U$ an (open) neighborhood of the diagonal. For $x \in M$, let $C(x)$ be the set of all $y \in M$ such that there is a $U$-chain from $x$ to $y$. Since $U[y] = \{z : (y, z) \in U\}$ is a neighborhood of $y$, it easily follows that $C(x)$ is open and closed. So $C(x) = M$ and $M$ is $U$-chainable.

Suppose $M$ is disconnected and $M = A \cup B$ is a disconnection of $M$. Then $U = (A \times A) \cup (B \times B)$ is an open neighborhood of diagonal, and since $U$ is transitive ($U \circ U = U$) for $x \in A$ and $y \in B$, there is no $U$-chain from $x$ to $y$. This proves the lemma.

Note that when $M$ is metrizable, then $M$ is connected iff $M$ is $(d, \varepsilon)$-chainable for each $\varepsilon > 0$ and $d \in D$.

Our next lemma is an extension of Banach's contraction principle for the class of connected, metrizable spaces.*

Lemma 2. *Let $M$ be a connected and metrizable topological space and $f: M \to M$ continuous. If there is a $d \in D(M)$ such that $(M, d)$ is complete and $f$ satisfies (SLC), then all iterates converge to a unique fixed point of $f$.*

Proof. For $x \in M$, let

$$r(x) = \sup\{r : f \text{ satisfies } (SLC) \text{ on } S(x, r)\}.$$

If for some $x \in M$; $r(x) = +\infty$, then $f$ is a strong contraction on the whole space $M$ and the lemma will follow from the Banach's contraction principle. So we assume that $r(x) < +\infty$ for each $x \in M$. Let

$$V = \{(x, y) : d(x, y) < {}^1/_2 \min(r(x), r(y))\}.$$

If $\Delta$ is the diagonal in $M \times M$, then evidently $\Delta \subset V$. We prove now that $\operatorname{int} V \supset \Delta$. Suppose the contrary that $\operatorname{int} V \not\supset \Delta$. Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_n \to x_0, \; y_n \to x_0 \text{ and } (x_n, y_n) \notin V,$$

what also means that

$$d(x_n, y_n) \geq {}^1/_2 \min\{r(x_n), r(y_n)\}, \quad n = 1, 2, \ldots \tag{1}$$

Let $n_1$ be such a natural number that

$$d(x_n, y_n) < r(x_0)/4, \quad \text{for } n \geq n_1.$$

Then we also have

$$x_n \text{ and } y_n \in S(x_0, r(x_0)/2), \quad \text{for } n \geq n_2. \tag{2}$$

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* (an interstep, not a standard generalization)

From (2) it follows that

$$r(x_n) \geq r(x_0)/2 \text{ and } r(y_n) \geq r(x_0)/2, \text{ for } n \geq n_0 = \max\{n_1, n_2\}.$$

So for an arbitrary $n \geq n_0$,

$$d(x_n, y_n) > {}^1/_2 \, r(x_0)/2 \leq {}^1/_2 \min\{r(x_n), r(y_n)\},$$

what contradicts (1). Hence $\Delta \subset \operatorname{int} V$.

Now let $(u, v) \in \operatorname{int} V = U$. Since $v \in S(u, r(u)/2)$ and $f$ satisfies $(SLC)$ on $S(u, r(u)/2)$, we have

$$(3) \qquad d(f^n u, f^n v) \leq q^n K(u, v), \quad n = 1, 2, \ldots$$

Let $x \in M$. Since $M$ is connected, according to Lemma 1, $M$ is $U$-chainable for $U = \operatorname{int} V$ and let $x, z_1, \ldots, z_n, fx$ be a $V$-chain from $x$ to $fx$. Using (3) and applying the triangle inequality, we get

$$d(f^n x; f^{n+1} x) \leq q^n [K(x, z_1) + \cdots + K(z_n, y)].$$

This implies that $\{f^n x\}$ is a Cauchy sequence. Since $(M, d)$ is complete, it follows that $f^n x \to \tilde{x}$ and the continuity of $f$ implies $f\tilde{x} = \tilde{x}$.

If $\tilde{y}$ is also a fixed point of $f$, then taking a $U$-chain

$$\tilde{x}, u_1, \ldots, u_m, \tilde{y},$$

from $\tilde{x}$ to $\tilde{y}$, we obtain

$$d(\tilde{x}, \tilde{y}) \leq q^n \{K(\tilde{x}, u_1) + \cdots + K(u_m, \tilde{y})\},$$

for every $n$. Hence $d(\tilde{x}, \tilde{y}) = 0$ or $\tilde{x} = \tilde{y}$. This proves the uniqueness of $\tilde{x}$ and concludes the proof of Lemma 2.

Lemma 3. *Let $(M, d)$ be a metric space connected by finite arcs. If $f:(M, d) \to (M, d)$ satisfies $(LC)$ then $f$ also satisfies $(SLC)$.*

Proof. Let $S: 0 < t_1 \ldots < t_n = 1$ be a subdivision of the interval $[0, 1]$ and

$$\| S \| = \sup\{t_{i+1} - t_i : i = 0, 1, \ldots, n-1\}.$$

Denote the sequence

$$f^k(h(t_0)), \; f^k(h(t_1)), \; f^k(h(t_n)),$$

where $h:[0, 1] \to L$ is a homeomorphism onto a finite arc $L$ from $x$ to $y$, by $f^k \circ h(S)$.

Let

$$V = \{(x, y) : d(fx, fy) \leq q(x, y)\},$$

and put $U = \operatorname{int} V$. Repeating the part of the proof of Lemma 2, it follows that $\Delta \subset \operatorname{int} V$.

Now we prove that for each $k$ there is an $\varepsilon_k > 0$ such that for every $S$ for which $\|S\| < \varepsilon_k$, the sequence $f^k \circ h(S)$ is a $U$-chain from $x$ to $y$. Indeed, supposing the contrary, we would have two sequences $\{t_n\}$ and $\{t_n'\}$ such that $t_n \to t_0$, $t_n' \to t_0$, and that

(4) $$(f^k h(t_n), f^k h(t_n')) \notin U$$

The mapping $f^k \circ h$ is so continuous,

$$x_n = f^k \circ h(t_n) \to x_0, \; y_n = f^k \circ h(t_n') \to x_0,$$

where $x_0$ is the element in $M$ such that $x_0 = f^k \circ h(t_0)$. Since $U$ is open, this would imply

(5) $$(x_n, y_n) \in U, \text{ for } n \geq \tilde{n},$$

where $\tilde{n}$ is a suitably chosen natural number. But (5) contradicts (4) and so $f^k \circ h(S)$ is a $U$-chain for $\|S\| < \varepsilon_k$. Let $n$ be fixed. Then for each $k = 0, 1, \ldots, n$ there is an $\varepsilon_k$ such $f^k \circ h(S)$ is a $U$-chain for $\|S\| < \varepsilon_k$. Let $\varepsilon = \min\{\varepsilon_k : k = 0, 1, \ldots, n\}$, then for an $S: 0 = t_0 < t_1 \ldots < t_m = 1$ such that $\|S\| < \varepsilon$, we have

$$d(f^n x, f^n y) \leq \sum \{d(f^n \circ h(t_i), f^n \circ h(t_{i+1}) : i = 0, \ldots, m-1\} < q^n \|L\|,$$

what was to be proved.

These three lemmas establish our theorem.

Note that in a part of the proof of Lemma 2 (and Lemma 3.) we had to show that the covering $\{S(x, r(x)) : x \in M\}$ is even ([2], p. 155).

**4. Corollaries.** The first corollary is an immediate consequence of the above theorem.

Corollary 1. *Let $C$ be a closed, convex subset of a Banach space and $f: C \to C$ satisfies (LC). Then $f$ has a unique fixed point to which all iterates converge.*

Corollary 2. *Let $M$ be a connected, metrizable topological space and $d \in D(M)$ such that $f: (M, d) \to (M, d)$ satisfies (LC). Then $f$ has a unique fixed point to which all iterates converge.*

Proof. Let $U = \operatorname{int} V$, where $V$ is again the set

$$\{(x, y) : d(fx, fy) \leq q\, d(x, y)\}.$$

Since $V$ is a neighborhood of the diagonal and $M$ is compact, there exists an $\varepsilon > 0$ such that

$$W = \{(x. y) : d(x, y) < \varepsilon\} \subseteq U.$$

According to Lemma 1, $M$ is $W$-chainable and $f$ is a $W$-contraction (evidently $fW \subseteq W$) and so $f$ satisfies ($SLC$). Now Lemma 2 implies this corollary.

Remarks. 1. The contents of this article, under the title "A contraction type theorem", were presented at the 4th Balkan Mathematical Congress. Istanbul, 1971 (Abstracts, 172). See also: Đ. Kurepa, "Some cases in the fixed point theory", Topology and its Applications, Beograd, 1973, 148 – 156.

2. There exist a metric on the real line $R$ (under which $R$ is not connected by finite arcs) and a local contraction $f$ on $R$ into itself having no fixed point.

## REFERENCES

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