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LIE THEORETIC GENERATING FUNCTIONS

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In this paper we propose to use the idea of Lie theory as suggested by Miller [2] and Weisner [4, 5, 6] with a view to obtaining generating functions. The process, in short, involves introducing first order linear differential operators generating a Lie Algebra isomorphic to sl (2) [2, p. 8], and then, based on these operators, determining a multiplier representation $[T(g)f](x, y)$, $g\in SL(2)$. By choosing $f(x, y)$ in a certain way, this multiplier representation leads us to generating functions.

2. Differential operators and multiplier representation

$$u(x)={}_1F_1\left[\begin{matrix}\lambda+n; & \\ & x\\ \gamma & \end{matrix}\right] \tag{2.1}$$

is a solution of [3, p. 124]

$$x\frac{d^2u}{dx^2}+(\gamma-x)\frac{du}{dx}-(\lambda+n)u=0. \tag{2.2}$$

We contsruct a partial differential equation by substituting $y\dfrac{\partial}{\partial y}$ for n, and, thus, have

$$\left[x\frac{\partial^2}{\partial x^2}+(\gamma-x)\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}-\lambda\right]f(x, y)=0. \tag{2.3}$$

$f(x, y)=y^n u(x)$ is a solution of (2.3).

Now we introduce the first order partial differential operators

$$J^3=y\frac{\partial}{\partial y}+\lambda-\frac{\gamma}{2},\quad J^+=xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+\lambda y, \tag{2.4}$$

$$J^-=xy^{-1}\frac{\partial}{\partial x}-\frac{\partial}{\partial y}+(\gamma-\lambda-x)y^{-1}$$

obeying the commutation relations

$$(2.5)\qquad [J^3,\ J^{\pm}]=\pm J^{\pm},\quad [J^+,\ J^-]=2J^3.$$

These J-operators form the basis of a Lie algebra isomorphic to the Lie algebra sl (2) [2, p. 8].

The Casimir operator

$$(2.6)\qquad C=J^+J^-+J^3J^3-J^3$$
$$=x^2\frac{\partial^2}{\partial x^2}+x(\gamma-x)\frac{\partial}{\partial x}-xy\frac{\partial}{\partial y}-\lambda x+\frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)$$

commutes with J^3, J^+ and J^-.

(2.3) may be rewritten as

$$(2.7)\qquad Cf(x,\ y)=\frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)f(x,\ y).$$

To determine the multiplier representation induced by J-operators, we need to compute the expression

$$(2.8)\qquad e^{a'J^-}e^{b'J^+}e^{c'J^3}.$$

The action of the group element $e^{a'J^-}$ is obtained by solving the differential equation [2, p. 18]

$$\frac{dx(a')}{da'}=\frac{x(a')}{y(a')},\quad \frac{dy(a')}{da'}=-1,$$

$$\frac{dv(a')}{da'}=v(a')\frac{\gamma-\lambda-x(a')}{y(a')},\quad x(0)=x,\quad y(0)=y,\quad v(0)=1,$$

giving

$$x(a')=\frac{xy}{y-a'},\quad y(a')=y-a',$$

$$v=\left(1-\frac{a'}{y}\right)^{\lambda-\gamma}e^{\frac{-a'x}{y-a'}}.$$

Therefore

$$(2.9)\qquad e^{a'J^-}f(x,\ y)=e^{\frac{-a'x}{y-a'}}\left(1-\frac{a'}{y}\right)^{\lambda-\gamma}f\left(\frac{xy}{y-a'},\ y-a'\right)\quad\left|\frac{a'}{y}\right|<1.$$

Similarly

$$(2.10)\qquad e^{b'J^+}f(x,\ y)=(1-b'y)^{-\lambda}f\left(\frac{x}{1-b'y},\ \frac{y}{1-b'y}\right),\quad |b'y|<1$$

and

$$(2.11)\qquad e^{c'J^3}f(x,\ y)=e^{\left(\lambda-\frac{\gamma}{2}\right)c'}f(x,\ ye^{c'}).$$

Thus

$$e^{a'J^-}e^{b'J^+}e^{c'J^3}=e^{\left(\lambda-\frac{\gamma}{2}\right)c'}e^{\frac{-a'x}{y-a'}}\left(1-\frac{a'}{y}\right)^{\lambda-\gamma}\times \tag{2.12}$$

$$(1+a'b'-b'y)^{-\lambda}f\left(\frac{xy}{(y-a')(1+a'b'-b'y)},\ \frac{y-a'}{1+a'b'-b'y}e^{c'}\right).$$

The complex parameters a', b' and c' are related to $g\in SL(2)$,

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix},\quad ad-bc=1,$$

by [2, p. 8]

$$e^{c'/2}=a,\quad a'=-\frac{c}{a},\quad b'=-ab.$$

Therefore, for g in a sufficiently small neighbourhood of the identity element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\in SL(2)$

$$[T(g)f](x,y)=e^{\frac{cx}{c+ay}}\left(a+\frac{c}{y}\right)^{\lambda-\gamma}(d+by)^{-\lambda}\times \tag{2.13}$$

$$f\left(\frac{xy}{(c+ay)(d+by)},\ \frac{c+ay}{d+by}\right),$$

$$\left|\frac{c}{ay}\right|<1,\quad \left|\frac{by}{d}\right|<1,\quad -\pi<\arg a,\quad \arg d<\pi,\quad ad-bc=1.$$

3. Generating functions

We choose $f(x,y)$ to be a common eigen function of the operators C and $J^3J^3+(\gamma+\gamma'-2\lambda-1)J^3-J^+$.

Let $f(x,y)$ satisfy the simultaneous equations

$$Cf(x,y)=\frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)f(x,y), \tag{3.1}$$

$$[J^3J^3+(\gamma+\gamma'-2\lambda-1)J^3-J^+]f(x,y)$$

$$=\left(\lambda-\frac{\gamma}{2}\right)\left(\frac{\gamma}{2}+\gamma'-\lambda-1\right)f(x,y)$$

which may be rewritten as

$$\left[x\frac{\partial^2}{\partial x^2}+(\gamma-x)\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}-\lambda\right]f(x, y)=0, \tag{3.2}$$

$$\left[y\frac{\partial^2}{\partial y^2}+(\gamma'-y)\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}-\lambda\right]f(x, y)=0.$$

These equations have a solution [1, p. 234]

$$f(x, y)=\psi_2[\lambda;\ \gamma, \gamma';\ x, y] \tag{3.3}$$

where

$$\psi_2[\lambda;\ \gamma, \gamma';\ x, y]=\sum_{m, n=0}^{\infty}\frac{(\lambda)_{m+n}}{m!\, n!\, (\gamma)_m (\gamma')_n}\, xy^n.$$

Therefore

$$[T(g)f](x, y)=e^{\frac{cx}{c+ay}}\left(a+\frac{c}{y}\right)^{\lambda-\gamma}(d+by)^{-\lambda}\times \tag{3.4}$$

$$\psi_2\left[\lambda;\ \gamma, \gamma';\ \frac{xy}{(c+ay)\,(d+by)},\ \frac{c+ay}{d+by}\right],$$

$$\left|\frac{c}{ay}\right|<1,\quad \left|\frac{by}{d}\right|<1,\quad -\pi<\arg a,\quad \arg d<\pi,\quad ad-bc=1.$$

$[T(y)f]\,(x, y)$ satisfies

$$C[T(g)f](x, y)=\frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)[T(g)f](x, y). \tag{3.5}$$

(3.4) has an expansion of the form

$$[T(g)f](x, y)=\sum_{n=-\infty}^{\infty} j_n(g)\, {}_1F_1\left[\begin{matrix}\lambda+n; \\ \\ \gamma\end{matrix}\ x\right]y^n. \tag{3.6}$$

Putting $x=0$, this gives

$$j_n(g)=(-1)^n a^{\lambda-\gamma} d^{-\lambda-n} b^n \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\times \tag{3.7}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda+n)_r\,(\gamma-\lambda)_r}{r!\,\Gamma(1+n+r)}\left(\frac{bc}{ad}\right)^r$$

$${}_2F_2\left[\begin{matrix}-n-r, & 1+\lambda-r; \\ & \\ \gamma', & 1+\lambda-\gamma-r\end{matrix}\ \frac{a}{b}\right].$$

Thus, the generating function (3.6) becomes

$$(3.8)\qquad e^{\frac{cx}{c+ay}}\left(1+\frac{c}{ay}\right)^{\lambda-\gamma}\left(1+\frac{by}{d}\right)^{-\lambda}\times$$

$$\psi_2\left[\lambda;\ \gamma,\gamma';\ \frac{xy}{(c+ay)(d+by)},\ \frac{c+ay}{d+by}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\left(-\frac{by}{d}\right)^n {}_1F_1\left[\begin{matrix}\lambda+n;\\ \gamma\end{matrix}\ x\right]\times$$

$$\sum_{r=0}^{\infty}\frac{(\lambda+n)_r(\gamma-\lambda)_r}{r!\,\Gamma(1+n+r)}\times$$

$$\left(\frac{bc}{ad}\right)^r {}_2F_2\left[\begin{matrix}-n-r,\ 1+\lambda-\gamma;\\ \gamma',\ 1+\lambda-\gamma-r;\end{matrix}\ \frac{a}{b}\right],$$

$$\left|\frac{c}{ay}\right|<1,\quad \left|\frac{by}{d}\right|<1,\quad ad-bc=1,$$

where the terms corresponding to $n=-1,-2,-3,\ldots$ are well defined because of the relation

$$(3.9)\qquad \operatorname*{Lt}_{\mu\to-k}\sum_{r=0}^{\infty}\frac{(\lambda+\mu)_r(\gamma-\lambda)_r}{r!\,\Gamma(1+\mu+r)}\left(\frac{bc}{ad}\right)^r$$

$${}_2F_2\left[\begin{matrix}-\mu-r,\ 1-\lambda-\gamma;\\ \gamma'\ \ 1+\lambda-\gamma-r;\end{matrix}\ \frac{a}{b}\right]$$

$$=\frac{(\lambda-k)_k(\gamma-\lambda)_k}{k!}\left(\frac{bc}{ad}\right)^k\sum_{r=0}^{\infty}\frac{(\lambda)_r(\gamma-\lambda+k)_r}{r!(1+k)_r}\left(\frac{bc}{ad}\right)^r\times$$

$${}_2F_2\left[\begin{matrix}-r,\ 1+\lambda-\gamma;\\ \gamma',\ 1+\lambda-\gamma-r-k;\end{matrix}\ \frac{a}{b}\right],$$

$$k=1,2,3,\ldots$$

(3.9) yields the following special cases:

(3.10) $$e^{\frac{wx}{w+y}}\left(1+\frac{w}{y}\right)^{\lambda-\gamma}\psi_2\left[\lambda;\ \gamma,\gamma';\ \frac{xy}{w+y},\ w+y\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\Gamma(\lambda+n)\Gamma(\gamma')}{\Gamma(1+n)\Gamma(\lambda)\Gamma(\gamma'+n)}\,{}_1F_1\left[\begin{matrix}\lambda+n;\\ \gamma\end{matrix}\ x\right]\times$$

$${}_2F_2\left[\begin{matrix}\lambda+n, & 1+\lambda-\gamma+n;\\ 1+n, & \gamma'+n;\end{matrix}\ w\right]y^n,\quad \left|\frac{w}{y}\right|<1;$$

(3.11) $$(1-y)^{-\lambda}\psi_2\left[\lambda;\ \gamma,\gamma';\ \frac{x}{1-y},\ \frac{-wy}{1-y}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_1F_1\left[\begin{matrix}\lambda+n;\\ \gamma;\end{matrix}\ x\right]{}_1F_1\left[\begin{matrix}-n;\\ \gamma';\end{matrix}\ w\right]y^n,$$

$$|y|<1.$$

In (3.10) the terms corresponding to $n=-1,-2,-3,\ldots$ are well defined in view of the relation

(3.12) $$\operatorname*{Lt}_{\mu\to-k}\ {}_2F_2\frac{\left[\begin{matrix}\lambda+\mu, & 1+\lambda-\gamma+\mu;\\ 1+\mu, & \gamma'+\mu;\end{matrix}\ w\right]}{\Gamma(1+\mu)}$$

$$=\frac{(\lambda-k)_k(1+\lambda-\gamma-k)_k}{k!(\gamma'-k)_k}w^k\,{}_2F_2\left[\begin{matrix}\lambda, & 1+\lambda-\gamma;\\ 1+k, & \gamma';\end{matrix}\ w\right],$$

$$k=1,2,3,\ldots$$

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