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ON THE ABSOLUTE NÖRLUND SUMMABILITY OF A SERIES ASSOCIATED WITH A FOURIER SERIES

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1. Let $\{p_n\}$ be a sequence of complex numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \neq 0, \quad P_{-1} = p_{-1} = 0.$$

The transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} S_\nu, \tag{1.1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$ of partial sums of the series $\sum a_n$. The series $\sum a_n$ is said to be summable (N, p_n) to the sum s if $\lim_{n\to\infty} t_n = s$, and the series is said to be summable $|N, p_n|$ to the sum s if in addition

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty. \tag{1.2}$$

In case when $\{p_n\} \in M$, that is,

$$p_n > 0 \text{ and } \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1,$$

(1.2) holds if and only if [4]

$$\sum_{n=1}^{\infty} \frac{1}{n P_n} \left| \sum_{\nu=1}^{n} p_{n-\nu} \nu a_\nu \right| < \infty.$$

A sequence $\{\mu_n\}$ is said to be a moment sequence if the $\{\mu_n\}$ are moments of a function $\chi(x)$ of bounded variation in the interval $0 \leqslant x \leqslant 1$,

$$\mu_n = \int_0^1 x^n \, d\chi(x), \quad n = 0, 1, 2, \ldots;$$

the function x^0 is defined at $x=0$ so as to be continuous. It is also supposed that $\chi(0)=0$. If also, $\chi(1)=1$ and $\chi(+0)=0$, so that $\chi(x)$ is continuous at origin, then μ_n is a regular moment constant.

The transformation

$$H_n=\sum_{\nu=0}^{n}\binom{n}{\nu}(\Delta^{n-\nu}\mu_\nu)\,S_\nu, \tag{1.3}$$

where

$$\Delta^0\mu_n=\mu_n \text{ and } \Delta^p\mu_n=\Delta^{p-1}\mu_n-\Delta^{p-1}\mu_{n+1},\quad p\geqslant 1,$$

defines the sequence $\{H_n\}$ of $(\mathfrak{H},\mu)$ means or the Hausdorff means of the sequence $\{s_n\}$. The series $\sum a_n$ is said to be $(\mathfrak{H},\mu)$ summable to the sum s if $\lim_{n\to\infty} H_n=s$, and is said to be $|\mathfrak{H},\mu|$ summable if

$$\sum_{n=1}^{\infty}|H_n-H_{n-1}|<\infty.$$

For the transformation $(\mathfrak{H},\mu)$ to be convergence preserving it is necessary and sufficient that μ_n is a moment constant. If the moment constant μ_n is regular, then the transformation $(\mathfrak{H},\mu)$ is also regular. A sequence to sequence Hausdorff transformation is absolute convergence preserving or absolutely regular if and only if it is a convergence preserving or regular transformation of the same type ([6], [14], [15]).

If in (1.1) we take

$$p_n=\binom{n+\alpha-1}{\alpha-1}=\frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)},\quad \alpha>0,$$

or in (1.3) we take

$$\chi(x)=1-(1-x)^\alpha,\quad \alpha>0,$$

the Nörlund and the Hausdorff means both reduce to the Cesàro mean (C,α) of order α.

2. Let $f(t)$ be a periodic function with period 2π and integrable in the Lebesgue sense in $(-\pi,\pi)$. Let the Fourier series associated with the function $f(t)$ be

$$f(t)\sim\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)\equiv\sum_{n=1}^{\infty}A_n(t).$$

We write

$$\varphi_{lc}(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}.$$

For $\alpha>0$, the α-th integral of the function $\Phi_{lc}(t)$ is defined

$$\varphi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}\Phi_{lc}(u)\,du.$$

For $\alpha=0$,

$$\varphi_0(t)=\varphi_{lc}(t).$$

Also,

$$\varphi_{lc\alpha}(t)=\Gamma(\alpha+1)\,t^{-\alpha}\varphi_\alpha(t),\quad \alpha\geqslant 0.$$

The p-th forward and backward fractional integrals of a function $g(x)$, which is Lebesgue integrable in (0,1), are respectively defined as

$$g_p^+(x)=\frac{1}{\Gamma(p)}\int_0^x (x-u)^{p-1}g(u)\,du,$$

and

$$g_p^-(x)=\frac{1}{\Gamma(p)}\int_x^1 (u-x)^{p-1}g(u)\,du.$$

These integrals exist almost everywhere for $p>0$.

We adopt the following notations:

$$N_n(t)\equiv\frac{2}{\pi}\sum_{k=1}^{n} p_{n-k}\,\varepsilon(k)\sin kt;$$

$$J(n,u)\equiv\frac{1}{\Gamma(1-\alpha)}\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}N_n(t)\,dt;$$

$$V(n,u)\equiv\frac{1}{\Gamma(1+\alpha)}\int_0^u v^\alpha\frac{d}{dv}J(n,v)\,dv.$$

$[x]$ denotes the integral part of x and C stands for an absolute constant not necessarily the same at each occurrence.

3. Taking the start from the work of Hille and Tamarkin [5], Astrachan [1] obtained the following result on the Nörlund summability of a Fourier series.

Theorem A. *If $\varphi_{lc\alpha}(t)\,(0<\alpha<1)=0(1)$ as $t\to 0$, and the sequence $\{p_n\}$ satisfies the conditions*

$$n|p_n|<C|P_n|,$$

$$\sum_{k=1}^{n} k\,|p_k-p_{k-1}|<C|P_n|,$$

$$\sum_{k=1}^{n} k(n-k)\,|p_k-2p_{k-1}+p_{k-2}|<C|P_n|,$$

$$\sum_{k=1}^{n}\frac{|P_k|}{k^2}<C\frac{|P_n|}{n},$$

then the Fourier series of $f(t)$ is summable (N,p_n).

The particular case $\varepsilon(t)=1$ of Theorem 1 which we establish in section 5 of this paper is an analogue of the above theorem for the absolute summability when the generating sequence $\{p_n\}\in M$. Theorem 2 which embodies a result on the absolute Cesàro summability of Fourier series generalises well-known results due to Bosanquet ([2] Theorem 1 and the case $0<\alpha<1$ of Theorem 1 in [3]), Mohanty ([12], [13]) and Matsumoto [9].

Theorem 1. *Let $\varepsilon(t)$ be a positive, monotonic non-decreasing function such that*

$$\int_0^{\pi} \varepsilon\left(\frac{r}{t}\right) |d\varphi_{lc\alpha}(t)| < C, \quad (r > \pi,\ 0 < \alpha < 1). \tag{3.1}$$

If $\{p_n\} \in M$, and

$$\sum_{n=N}^{\infty} \frac{\varepsilon(n)}{n^{1-\alpha} P_n} = O\left(\frac{\varepsilon(N) N^{\alpha}}{P_N}\right), \tag{3.2}$$

then the series $\sum A_n(t)\varepsilon(n)$ is summable $|N, p_n|$.

Theorem 2. *Let $\varepsilon(t)$ be a positive, monotonic non-decreasing function such that*

$$\sum_{n=N}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma-\alpha}} = O\left(\frac{\varepsilon(N)}{N^{\gamma-\alpha}}\right), \tag{3.3}$$

and

$$\int_0^{\pi} \varepsilon\left(\frac{r}{t}\right) |d\varphi_{lc\alpha}(t)| < C, \quad (0 \leqslant \alpha < \gamma < 1,\ r > \pi)$$

then the series $\sum A_n(t)\varepsilon(n)$ is summable $|C, \gamma|$.

4. The following lemmas are pertinent to the proof of Theorem 1.

Lemma 1. [11]. *If $\{p_n\}$ is a non-negative and non-increasing sequence, then, for $0 \leqslant a < b \leqslant \infty$, $0 < t \leqslant \pi$, and any n,*

$$\left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| \leqslant C P_{\left[\frac{1}{t}\right]}.$$

Lemma 2. *If $\{p_n\}$ is non-negative and non-increasing and $\{\varepsilon(n)\}$ is non-negative and non-decreasing, then*

$$N_n(t) = \begin{cases} O(\varepsilon(n) P_n) \text{ for all } t, \\ O\left(\varepsilon(n) P_{\left[\frac{1}{t}\right]}\right) \text{ for } t \geqslant \frac{1}{n}. \end{cases} \tag{4.1}$$

And

$$\frac{d}{dt} N_n(t) \equiv \begin{cases} O(n\varepsilon(n) P_n) \text{ for all } t, \\ O\left(n\varepsilon(n) P_{\left[\frac{1}{t}\right]}\right) \text{ for } t \geqslant \frac{1}{n}. \end{cases} \tag{4.2}$$

Proof. We have

$$|N_n(t)| \leqslant \frac{2}{\pi} \sum_{k=1}^{n} p_{n-k}\varepsilon(k),$$

and this gives the first part of the estimate in (4.1). The second part follows easily by the application of Lemma 1. The estimates in (4.2) can be similarly established.

Lemma 3. *For* $0<\alpha<1$,

$$J(n,u)=\begin{cases} O\left(n^{\alpha}\varepsilon(n)P_n\right) \text{ for all } u, \\ O\left(n^{\alpha}\varepsilon(n)P_{\left[\frac{1}{u}\right]}\right) \text{ for } u\geqslant \frac{1}{n}. \end{cases} \tag{4.3}$$

Proof. We have

$$J(n,u)=\frac{1}{\Gamma(1-\alpha)}\int\limits_{u}^{u+\frac{1}{n}}(t-u)^{-\alpha}\frac{d}{dt}N_n(t)\,dt+\frac{1}{\Gamma(1-\alpha)}\int\limits_{u+\frac{1}{n}}^{\pi}(t-u)^{-\alpha}\frac{d}{dt}N_n(t)\,dt=$$

$$=O(n\varepsilon(n))\int\limits_{u}^{u+\frac{1}{n}}(t-u)^{-\alpha}P_{\left[\frac{1}{t}\right]}\,dt+O\left(n^{\alpha}\left|\int\limits_{u+\frac{1}{n}}^{\eta}\frac{d}{dt}N_n(t)\,dt\right|\right)\left(u+\frac{1}{n}<\eta<\pi\right)$$

$$=O\left(n^{\alpha}\varepsilon(n)P_{\left[\frac{1}{u}\right]}\right),$$

by the application of the second parts of the estimates in (4.2) and (4.1). This establishes the second part of the estimate in (4.3). The first part can be similarly established if we use the first parts of the estimates in (4.2) and (4.1). Hence the lemma.

5. Proof of Theorem 1. Since $\{p_n\}\in M$, to prove the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{nP_n}\left|\sum_{\nu=1}^{n}p_{n-\nu}\varepsilon(\nu)\nu A_{\nu}(x)\right|<\infty. \tag{5.1}$$

Since

$$\nu A_{\nu}(x)=\frac{2}{\pi}\int\limits_{0}^{\pi}\varphi_{lc}(t)\frac{d}{dt}\sin\nu t\,dt$$

we have

$$\sum_{\nu=1}^{n}p_{n-\nu}\varepsilon(\nu)\nu A_{\nu}(x)=\int\limits_{0}^{\pi}\varphi_{lc}(t)\frac{d}{dt}N_n(t)\,dt$$

$$=\int\limits_{0}^{\pi}\frac{d}{dt}N_n(t)\left\{\frac{1}{\Gamma(1-\alpha)}\int\limits_{0}^{t}(t-u)^{-\alpha}\,d\Phi_{\alpha}(u)\right\}dt$$

$$= \int_0^{\pi} J(n,u)\, d\,\Phi_\alpha(u)$$

$$= \Phi_{lc\alpha}(\pi) J(n,\pi) - \varphi_\alpha(\pi) V(n,\pi) + \int_0^{\pi} V(n,u)\, d\,\varphi_{lc\alpha}(u)$$

$$= O\big(n^\alpha \varepsilon(n)\big) - \varphi_{lc\alpha}(\pi) V(n,\pi) + \int_0^{\pi} V(n,u)\, d\varphi_\alpha(u),$$

by the application of the second part of the estimate in (4.3) of Lemma 3.

If, in particular, we take $\varphi_{lc}(t)=1$, then $\varphi_{lc\alpha}(t)=1$ and $A_n(x)=0$ for $n=1,2,\dots,$ and therefore

$$\varphi_{lc\alpha}(\pi) V(n,\pi) = O\big(n^\alpha \varepsilon(n)\big).$$

Hence

$$\sum_{\nu=1}^{n} p_{n-\nu}\, \varepsilon(\nu)\, \nu A_\nu(x) = O\big(n^\alpha \varepsilon(n)\big) + \int_0^{\pi} V(n,u)\, d\varphi_{lc\alpha}(u),$$

and therefore

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{\nu=1}^{n} p_{n-\nu}\, \varepsilon(\nu)\, \nu A_\nu(x) \right|$$

$$\leqslant C \sum_{n=1}^{\infty} \frac{\varepsilon(n)}{n^{1-\alpha} P_n} + C \int_0^{\pi} |d\varphi_{lc\alpha}(u)| \sum_{n=1}^{\infty} \frac{1}{nP_n} |V(n,u)|$$

$$\leqslant C + C \int_0^{\pi} \varepsilon\left(\frac{r}{u}\right) |d\varphi_{lc\alpha}(u)| < \infty.$$

provided that

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} |V(n,u)| \leqslant C\varepsilon\left(\frac{r}{u}\right), \tag{5.2}$$

uniformly for all u $(0<u\leqslant\pi)$.

Thus to prove (5.1) we have to show that (5.2) holds.

Write

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} |V(n,u)| = \sum_{n=1}^{\left[\frac{r}{u}\right]} \frac{1}{nP_n} |V(n,u)| + \sum_{n=\left[\frac{r}{u}\right]+1}^{\infty} \frac{1}{nP_n} |V(n,u)| = \sum\nolimits_1 + \sum\nolimits_2. \tag{5.3}$$

Now

$$V(n,u) = \frac{u^\alpha J(n,u)}{\Gamma(1+\alpha)} - \frac{1}{\Gamma(\alpha)} \int_0^u v^{\alpha-1} J(n,v)\, dv$$

$$= O\big(n^\alpha u^\alpha \varepsilon(n) P_n\big),$$

by the application of the first estimate in (4.3) of Lemma 3. And therefore

$$(5.4)\qquad \sum\nolimits_1 = O\left(u^\alpha \varepsilon\left(\frac{r}{u}\right)\right) \sum_{n=1}^{\left[\frac{r}{u}\right]} n^{\alpha-1} = O\left(\varepsilon\left(\frac{r}{u}\right)\right).$$

The second estimate in (4.3) of Lemma 3 gives for $u \geqslant \frac{1}{n}$,

$$V(n,u) = V(n,\pi) - \left[\frac{v^\alpha J(n,v)}{\Gamma(1+\alpha)}\right]_u^\pi + \frac{1}{\Gamma(\alpha)} \int_u^\pi v^{\alpha-1} J(n,v)\,dv$$

$$= O\left(n^\alpha \varepsilon(n)\right) + O\left(n^\alpha u^\alpha \varepsilon(n) P_{\left[\frac{1}{u}\right]}\right) +$$

$$+ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_u^\pi v^{\alpha-1}\,dv \int_v^\pi (t-v)^{-\alpha} \frac{d}{dt} N_n(t)\,dt$$

$$= O\left(n^\alpha \varepsilon(n)\right) + O\left(n^\alpha u^\alpha \varepsilon(n) P_{\left[\frac{1}{u}\right]}\right)$$

$$+ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_u^\pi \frac{d}{dt} N_n(t)\,dt \int_{\frac{u}{t}}^1 (1-\xi)^{-\alpha} \xi^{\alpha-1}\,d\xi$$

$$= O\left(n^\alpha \varepsilon(n)\right) + O\left(n^\alpha u^\alpha \varepsilon(n) P_{\left[\frac{1}{u}\right]}\right),$$

by the application of the second estimate in (4.1) of Lemma 2.

Using the above estimate of $V(n,u)$, we get

$$(5.5)\qquad \sum\nolimits_2 = O(1) \sum_{n=1}^\infty \frac{\varepsilon(n)}{n^{1-\alpha} P_n} + O\left(u^\alpha P_{\left[\frac{1}{u}\right]}\right) \sum_{n=\left[\frac{r}{u}\right]+1}^\infty \frac{\varepsilon(n)}{n^{1-\alpha} P_n}$$

$$= O(1) + O\left(\varepsilon\left(\frac{r}{u}\right)\right) = O\left(\varepsilon\left(\frac{r}{u}\right)\right).$$

Combining the estimates in (5.4) and (5.5) we find, in view of (5.3), that (5.2) is established and this completes the proof of Theorem 1.

Proof of Theorem 2. The case $\alpha = 0$ of this theorem is known [10] and the case $0 < \alpha < 1$ is a particular case of Theorem 1.

6.* In a recent paper the author and Siya Ram [8] established the following

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* The author is grateful to Professor B. Kuttner for the contents of this section.

Theorem B. *Let* $\varepsilon(t)$ *be a positive and monotonic non-decreasing function of* t *such that*

$$(6.1) \qquad \sum_{n=\left[\frac{1}{t}\right]+1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma-\alpha}} = O\left(t^{\gamma-\alpha}\,\varepsilon\left(\frac{r}{t}\right)\right) \quad (0<t<\pi, \quad r<\pi),$$

and

$$(6.2) \qquad \int_0^{\pi} \varepsilon\left(\frac{r}{t}\right) |d\varphi_{Ic\alpha}(t)| < C.$$

If

$$\text{either} \quad \text{(a)} \ \chi(u) = g_{1+\gamma}^{+}(u) + C,$$

$$\text{or} \quad \text{(b)} \ \chi(u) = g_{1+\gamma}^{-}(u) + C,$$

for some function $g(u)$ *which is Lebesgue integrable in* $(0,1)$, *then the series* $\sum A_n(t)\,\varepsilon(n)$ *is summable* $|\mathfrak{H}, \mu|$ *at the point* $t=x$ *it being assumed that the transformation* $(\mathfrak{H}, \mu)$ *is convergence preserving.*

In this section we record that Theorem 2 and Theorem B are equivalent. In order to prove the equivalence we need the following lemma.

Lemma 4. *Let* $a_n>0$. *If*

$$\sum_{n=N}^{\infty} \frac{a_n}{n} = O(a_N),$$

then for sufficiently small $\eta>0$,

$$\sum_{n=N}^{\infty} a_n n^{\eta-1} = O(a_N N^{\eta}).$$

Proof. There exists a constant C such that for all $N \geqslant 1$,

$$(6.3) \qquad \sum_{n=N}^{\infty} \frac{a_n}{n} \leqslant C\,a_N.$$

Write

$$\chi_N = \sum_{n=N}^{\infty} \frac{a_n}{n}.$$

Then

$$a_N = N(\chi_N - \chi_{N+1}).$$

Replacing N by n, we find that (6.3) may be written in the form

$$\chi_n \leqslant C n (\chi_n - \chi_{n+1}),$$

and therefore

$$\chi_{n+1} \leqslant \chi_n \left(1 - \frac{1}{Cn}\right).$$

It follows immediately from the above inequality that

$$\chi_{2n} \leqslant \chi_n \prod_{\nu=n}^{2n-1} \left(1 - \frac{1}{C\nu}\right).$$

Since

$$\log \prod_{\nu=n}^{2n-1} \left(1 - \frac{1}{C\nu}\right) \to -\frac{1}{C} \log 2, \tag{6.4}$$

as $n \to \infty$, it follows that the expression on the left hand side of (6.4) is less than or equal to some negative constant for all $n \geqslant 1$. Hence there is a constant $K < 1$ such that

$$\chi_{2n} \leqslant K \chi_n \tag{6.5}$$

for all $n \geqslant 1$. Now, for $\eta > 0$ and $N \geqslant 1$,

$$\begin{aligned}\sum_{n=N}^{\infty} a_n n^{\eta-1} &= \left(\sum_{n=N}^{2N-1} + \sum_{n=2N}^{4N-1} + \cdots\cdot\right) a_n n^{\eta-1} \\ &\leqslant (2N)^{\eta} \sum_{n=N}^{2N-1} \frac{a_n}{n} + (4N)^{\eta} \sum_{n=2N}^{4N-1} \frac{a_n}{n} + \cdots\cdot \\ &\leqslant (2N)^{\eta} \chi_N + (4N)^{\eta} \chi_{2N} + (8N)^{\eta} \chi_{4N} + \cdots\cdot \\ &\leqslant N^{\eta} \chi_N [2^{\eta} + 4^{\eta} K + 8^{\eta} K^2 + \cdots\cdot]\end{aligned}$$

by virtue of the inequality in (6.5). If η is chosen small enough such that $2^{\eta} K < 1$, then the sum within the square brackets converges. Using (6.3) in the above inequality the lemma follows.

We now prove the equivalence. It is easy to verify that the conditions (3.3) and (6.1) are equivalent. Applying Lemma 4 with $a_n = \frac{\varepsilon(n)}{n^{\gamma-\alpha}}$ we find that if (3.3) holds, then it will still hold with γ replaced by $\gamma' < \gamma$ provided that $\gamma - \gamma'$ is sufficiently small. Assume first that Theorem 2 holds. Suppose that the hypotheses of Theorem B are satisfied; then they are still satisfied with γ replaced by $\gamma' < \gamma$ provided that $\gamma - \gamma'$ is sufficiently small. Hence applying Theorem 2 with γ replaced by γ' it follows that the series $\sum A_n(t)\, \varepsilon(n)$ is summable $|C, \gamma'|$. But it is known [7] that the summability $|C, \gamma'|$ implies summability $|\mathfrak{H}, \mu|$ under the hypotheses of Theorem B; hence Theorem B follows. Conversely assume that Theorem B holds. Again, if the hypotheses are satisfied they are still satisfied with γ replaced by $\gamma' < \gamma$ if $\gamma - \gamma'$ is sufficiently small. It is known (see [16] § 6) that the summability (C, γ) satisfies either of the conditions (a) and (b) with γ replaced by γ'. Hence applying Theorem B with γ replaced by γ', Theorem 2 follows.

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