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ON THE SYSTEM OF ALL MAXIMAL CHAINS (ANTICHAINS) OF A GRAPH.* 1)

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To my dear colleague and friend Erdös, Paul

0. Ordered sets constitute an important kind of graphs; it is a very natural way to transfer notions and results concerning ordered sets to general graphs.

0 – 1. From the very beginning of our study of ordered sets $(E, \leq)$ we stressed the importance of the number

(0 – 1 – 1) $\quad p_s(E, \leq) := \sup_A |A|,\ A \in a(E, \leq)$ where

(0 – 1 – 2) $\ a(E, \leq) := \{X \mid X \subset E;\ X$ is an antichain in $(E, \leq)\}$ [cf Kurepa 1935] p. 1196 and p. 1197, la relation fondamentale $(1) |E| \leq (2 p_s E)^{p_0 E}$]. The number $p_s(E, \leq)$ is called the *liberty degree of* $(E \leq)$ (s is initial of slav words *sloboda* or *svoboda* meaning liberty, freedom).

0 – 1 – 3. The question was whether the number $p_s E$ called also bridth of $(E, \leq)$ is *reached* i.e. whether the family

(0 – 1 – 4) $\ a_M(E, \leq) := \{X \mid X$ is a maximal antichain of $(E, \leq)$

has a maximum member-one of the greatest cardinality, i.e. the cardinality $p_s(E, \leq)$.

0 – 1 – 5. Obviously, for any graph (G, ρ) the corresponding notions $p_s(G, \rho)$, $a(G, \rho)$ $[a_M(G, \rho)]$ are defined in the same way and are called the independence number, the system of all [maximal] independent subsets.

0 – 2. The systems

(0 – 2 – 1) $\ L(E, \rho)$, $L_M(E, \rho)$ of all chains resp. of all maximal chains in (E, ρ) are well defined; for the case of graphs one speaks often of *complete subgraphs* instead of subchains in the graph.

0 – 2 – 2. R e m a r k. For every ordered set $(E, \leq)$ the empty set $\varnothing$ is considered as a subchain as well as an subantichain; in other words, $\varnothing$ is member of $L(E, \leq)$ as well as of $a(E, \leq)$.

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1) Ovaj rad je financirala *Republička zajednica za nauku SR Srbije* preko Matematičkog instituta.

0 – 3. Paths.

0 – 3 – 1. Definition. A *path* or *way* in graph is every well-ordered subset $W=(W_0, W_1, ..)$ such that for every W_β, $\beta>0$ the set of indices α such $\alpha<\beta$ and $W_\alpha \rho W_\beta$ is cofinal to the set of ordinals $<\beta$; thus in particular if $\beta^-<\beta$, then $W_{\beta^-} \rho W_\beta$.

0 – 3 – 2. Let $\pi(G, \rho)$ [resp. $\pi_M(G, \rho)$] be the system of all paths [resp. of all *maximal* paths] in (G, ρ).

0 – 3 – 3. Let $l(G, \rho):=\sup|X|$; the number $l(G, \rho)$ is called *path-length* of the graph.

0 – 3 – 4. Analogously the number $L_l(G, \rho):=\sup|X|$, $X\in L(G, \rho)$ may be called the chain-length of the graph.

0 – 4. Cellularity. For any system S of sets we define the cellularity cS or $\operatorname{cel} S$ as $cS:=\sup\limits_H |H|$, H running through the system of all disjoint subsystems of S; in particular, for any topological space E we define $cS:=c\,(GS)$, GS denoting the system of all open sets of the space E [cf. Kurepa [1935] p. 131 where cS was denoted by $p_2 S$.

0 – 5. As always when a supremum is concerned one has to examine whether the number L_l [resp. l] is reached if the graph is transfinite. The same question applies for the cellularity.

1. Cellularity of ordered chains

1 – 1. If $(C, \leq)$ is an ordered chain, then one could consider a *complete subdivision* of $(C, \leq)$ and get a corresponding tree $(T, \supset)$ of subintervals; one proves readily that

1 – 1 – 1. $\operatorname{cel}(C, \leq)=\operatorname{cel}(T, \supset)$ (cf. *Thèse* § 12; for instance Lemme 4 p. 121).

1 – 1 – 2. On the other hand, for every tree $(T, \leq)$ we defined a number $b'T$ as the supremum of cardinals $|F|$, F running through the system of all families of non radial elementary directions in $(T, \leq)$ (cf *Thèse* p. 109 § 4].

1 – 1 – 3. Now, if the rank or height $\gamma(T, \leq)$ is not cofinal to an inaccesible ordinal, then the number $b'(T, \leq)$ is reached (v. *Thèse* p. 110 Théorème 3). As an obvious corollary of this théorème 3 we have the following result.

1 – 1 – 4. Theorem. *If $(C, \leq)$ is any ordered chain such that the cellurarity $c(C, \leq)$ is not cofinal to an inaccessible number, then the cellularity $c(C, \leq)$ is reached, i.e. the chain $(C, \leq)$ contains a disjoint system of cardinality* $\operatorname{cel}(C, \leq)$ *of intervals of $(C, \leq)$. i.e. in the graph $(GC, X\cap Y=\varnothing)$ there is a maximum chain.*

The theorem 1 – 1 – 4 *should be compared to the following.*

1 – 1 – 5 Theorem. *The cellularity of squares of ordered chains is reached and is equal to the separability number of the chain (this result is implicitly contained in our papers* [1950], [1952].

As a matter of fact, it is sufficient to consider any complete bipartition D of $(C, \leq)$ (*Thèse* p. 114); if ψD denotes all elements of D of cardinality >1 each, then $(\psi D, \supset)$ is a tree of intervals in which every $X \in \psi D$ has two immediate followers X_0, X_1 such that $X_0 \leq X_1$; one has the corresponding rectangle $X_1 \times X_0$ in $(C^2, \leq)$; X running through ψD, the corresponding interiors int $X_1 \times X_0$ are open $\neq \varnothing$ and constitute a disjoint family H of cardinality $|\psi D|$; since $|\psi D| = \mathrm{sep}(C, \leq)$ one infers that

(1 – 1 – 5 – 1) $|H| = \mathrm{sep}(C, \leq)$.

On the other hand, obviously

(1 – 1 – 5 – 2) $|H| \leq \mathrm{cel}(C^2, \leq) \leq \mathrm{sep}(C, \leq)$; therefore we conclude that H is a maximum antichain in the graph $(G(C, \leq), \neq \varnothing)$ and this completes the proof oI the theorem 1 – 1 – 5.

1 – 1 – 6. As it was pointed out in *Thèse* (cf. *Principe de réduction* P_2, p. 130) the proposition $|T| = s(T, \leq) \cdot L_l(T, \leq)$ for infinite trees is a postulate; therefore we conclude that the proposition

1 – 1 – 7. *Every infinite ordered chain* $(C, \leq)$ *satisfies*

(1 – 1 – 8) $\mathrm{cel}(C^2, \leq) = \mathrm{cel}(C, \leq)$

is a postulate independant of other axioms in the ZF – set theory (cf. Kurepa [1974] for references).

2. Trees $\mathcal{T}_n (n = 0, 1, 2, \ldots)$.

$\mathcal{T}_0$ is the empty sequence; if $n \in N$, let T_n be composed of the empty sequence and of all elements of the set

$\{a := (a_1, a_2, \ldots, a_j) \mid 1 \leq i \leq j \leq n,\ a_i \in \{0, 1, \ldots, i-1\}\}$; let $a \dashv b$ mean that a is an initial segment of b; then we have the tree $\mathcal{T}_n := (T_n, \dashv)$ with quite interesting properties. At first we have the following

2 – 1. Theorem. *If* $n \in \{1, 2, \ldots\}$ *then*

(2 – 2) $$|a\mathcal{T}_n| = \underbrace{(\ldots((}_{n-1}2^n + 1)^{n-1} + 1)^{n-2} + \cdots + 1)^2 + 1)^1 + 1$$

$(2-2)_M$ $$|a_M\mathcal{T}_n| = \underbrace{(\ldots((}_{n-2}2^{n-1} + 1)^{n-2} + \cdots + 1)^2 + 1)^1 + 1.$$

Proof. $a\mathcal{T}_0 = \{\varnothing, \{\varnothing\}\}$; thus $|a\mathcal{T}_0| = 2$

$a\mathcal{T}_1 = \{\varnothing, \{\varnothing\}, \{(0)\}\}$; thus $|a\mathcal{T}_1| = 3 = |a_M\mathcal{T}_1|$.

Let $1 < n \in N$, then

(2 – 3) $a\mathcal{T}_n = a\mathcal{T}_1 \cup \{x \cup y \mid x \in [(0)_2, \cdot)_{\mathcal{T}_n},\ y \in a[(0, 1), \cdot)_{\mathcal{T}_n}\}$ (cf. § 4).

Now, if moreover $(x, y) \neq (x', y')$, then $x \cup y \neq x' \cup y'$. If $(x, y) = (\varnothing, \varnothing)$, then $x \cup y = \varnothing$. Thus $\varnothing$ is a common term – and unique one – of the two summands in (2 – 3). Therefore considering the cardinal numbers, the formula (2 – 3) yields

(2 – 4) $|a\mathcal{T}_n| = |a\mathcal{T}_1| + |a[(0, 0, .)_{\mathcal{T}_n}| \cdot |a[(0, 1), .)_{\mathcal{T}_n}| - 1.$ Thus

(2 – 5) $|a\mathcal{T}_n| = 2 + |a[(0, 0), .)_{\mathcal{T}_n}|^2$ because

$$|a[(0, 0), .)|_{\mathcal{T}_n} = |a[(0, 1), .)_{\mathcal{T}_n}|.$$

By a similar argument one proves

(2 – 6) $|a[(0, 0), .)_{\mathcal{T}_n}| = 1 + a[(0, 0, 0), .)_{\mathcal{T}_n}|^3,$

(2 – 7) $|a[0, 0, \ldots), .)_{\mathcal{T}_n}| = 1 + |a[(0)_i, .)_{\mathcal{T}_n}|^i$ for $2 \leq i \leq n.$[1]

In particular, for $i = n$ we have

(2 – 8) $|a[(0)_{n-1}, .)_{\mathcal{T}_n}| = 1 + |a[(0)_n, .)|^n.$

i.e.

(2 – 9) $|a[(0)_{n-1}, .)_{\mathcal{T}_n}| = 1 + 2^n$ because $a[(0)_n, .)_{\mathcal{T}_n} = \{\varnothing, \{(0)_n\}\}.$

The elimination of the intermediary terms yields

(2 – 10) $|a\mathcal{T}_n| = 2 + (1 + (1 + \cdots + (1 + 2^n)^{n-1})^{n-2} + \cdots)^3)^2,$ i.e.

we get the formula (2 – 2).

If we try to replace the symbol a by a_M in preceding formulas, then we see that we could do it in formulas (2 – 3) – (2 – 8). Only, instead of (2 – 9) we have

(2 – 11) $|a_M[(0)_{n-1}, .)_{\mathcal{T}_n}| = 1 + 1^n = 2,$

because $a_M[(0)_{n-1}, .)_{\mathcal{T}_n}$ consists of the singleton $\{(0)_n\}$. Finally, the formulas (2 – 11) and $(2 - 3) - (2 - 8)_M$ yield the requested equality $(2 - 2)_M$. This completes the proof of the theorem 2 – 1.

2 – 12. Remark. It is remarkable how the formulas (2 – 2), $(2 - 2)_M$ are tied: subindexing with M in $(2 - 2)_1$ (cf. § 4) implies replacing of 2^n by 1 in $(2 - 2)_2$; the removing of the index M in $((2 - 2)_M)_1$ implies the replacing of the basis 2 in the right part of $(2 - 2)_M$ by the expression $2^n + 1$.

3. The systems LT_n, $L_M T_n$. Let us prove the following

3 – 0. Theorem. *If* $n \in \{0, 1, 2, \ldots\}$, *then* (3 – 1) $|L_M T_n| = n!$

(3 – 2) $|LT_n| = 1 + \sum_{r=0}^{n} 2^r r!$

(3 – 3) $|LT_n| = |LT_{n-1}| + 2^n n!;$

in particular, we have the following table:

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[1] $(0)_i := (\underbrace{0, 0, \ldots, 0}_{i})$

n	0	1	2	3	4	5	6
2^n	1	2	4	8	16	32	64
$n!$	1	1	2	6	24	120	720
$2^n n!$	1	2	8	48	384	3840	46080
$\sum_{r=0}^{n} 2^r r!$	2	4	12	60	444	4284	50364

P r o o f o f t h e t h e o r e m. The proof of $(3-1)$ is immediate. Therefore let us prove $(3-2)$, $(3-3)$. At first since T_0 is the empty sequence $\varnothing$, we have

$$(3-4) \quad |L\mathcal{T}_0| = |\{\varnothing, \{\varnothing\}\} = 2.$$

3 – 4. Analogously

$$(3-5) \quad L\mathcal{T}_1 = \{\varnothing, \{\varnothing\}, \{(0)\}, \{\varnothing, (0)\}, |L\mathcal{T}_1| = 4.$$

In other words, the formula $(3-2)$ holds for $n=0, 1$. Since $(3-3)$, $(3-4)$ imply $(3-2)$, let us prove still $(3-3)$ for $n \in N$. Now,

$$(3-6) \quad L\mathcal{T}_n \setminus L\mathcal{T}_{n-1} = \{\{x\} \cup x' \mid x \in R_n \mathcal{T}_n, x' \subset \mathcal{T}_n(\cdot, x)\}.$$

Since x is of the form $x = (x_1, x_2, \ldots, x_n)$ where $x_i \in \{0, 1, \ldots, i-1\}$, x assumes $n!$ values i.e. $|R_n \mathcal{T}_n| = n!$. On the other hand, for every $x \in R_n \mathcal{G}_n$ the set

$$\mathcal{T}_n(\cdot, x) := \{\varnothing, (x_1), (x_1, x_2), \ldots, (x_1, x_2, \ldots, x_{n-1}\}$$

has n elements; therefore x' in $(3-6)$ assumes 2^n values; so does $\{x\} \cup x'$ as well: the system $(3-6)_2$ has just $2^n n!$ members; considering the cardinal numbers of $(3-6)_1$, $(3-6)_2$ we get precisely the requested formula $(3-3)$. Simple evaluations of $(3-2)$ for $n=0, 1, 2, 3, 4, 5, 6$ yielding the values indicated in the table, the theorem $(3-0)$ is completely proved.

3 – 7. Second proof of the theorem $(3-0)$. At first, $(3-4)$ holds; further, if $n \in N$, $L\mathcal{T}_n$ is formed of the chains obtained by adjoining to every member of $L\mathcal{T}_{n-1}$ a single member of $R_n \mathcal{T}_n$ (remark that $R_n \mathcal{T}_n$ has $n!$ members). Now to each member $x_1 \in L\mathcal{T}_1$ we adjoin each of the $n!$ members of $R_n \mathcal{T}_n$; to each member $x_2 \in L\mathcal{T}_2 \setminus L\mathcal{T}_1$ we can adjoin each of the $\frac{n!}{2!}$ members of $R_n \mathcal{T}_n$ following x_2; to each $0 < i < n$ and to every member $x_i \in L\mathcal{T}_i \setminus L\mathcal{T}_{i-1}$ we can adjoin each of the $\frac{n!}{i!}$ members of $R_n \mathcal{T}_n$ following x_i (as a matter of fact, $|R_i \mathcal{T}_n| = i!$ and each member of $R_i \mathcal{T}_i$ is followed by $\frac{n!}{i!}$ members of $R_n \mathcal{T}_n$). In other words

(3 – 8) $L\mathcal{T}_n = L\mathcal{T}_{n-1} \cup \{\{x, y\} \mid x \in L\mathcal{T}_0, y \in R_n \mathcal{T}_n[x]\} \cup$

$$\bigcup_{i=1}^{n-1} (\cup_{x_i, y_i} \{x_i \cup \{y_i\}\}, x_i \in L\mathcal{T}_i \setminus L\mathcal{T}_{i-1}, y_i \in R_n \mathcal{T}_n[x_i]).$$

Now, the constituting parts in (3 – 8) are pairwise disjoint; therefore passing to cardinalities and putting (3 – 9) $|L\mathcal{T}| = l_i (i = 0, 1, .)$ the formula 3 – 8 yields

$$l_n = l_{n-1} + l_0 n! + \sum_{i=1}^{n-1} \frac{(l_i - l_{i-1})}{i!} n!$$

and thus

$$(3-10) \quad \frac{l_n - l_{n-1}}{n!} = l_0 + \sum_{i=1}^{n-1} \frac{l_i - l_{i-1}}{i!}, \qquad (i = 2, 3, \ldots).$$

Put

$$(3-11) \quad q_i = \frac{l_i - l_{i-1}}{i!} \qquad (i = 1, 2, \ldots,);$$

thus in particular (cf. (3 – 4), (3 – 5)):

(3 – 12) $q_1 = l_1 - l_0 = 2, \; q_2 = 2^2.$

In virtue of (3 – 11) the relation (3 – 10) becomes

$$(3-13) \quad q_n = l_0 + \sum_{i=1}^{n-1} q_i \quad (n = 2, 3, \ldots).$$

Therefore

$$q_{n+1} - q_n = q_n, \text{ i.e.}$$

(3 – 14) $q_{n+1} = 2 q_n \qquad (n = 2, 3, \ldots)$ and consequently

$$q_{n+1} = 2^2 q_{n-1} = 2^3 q_{n-2} = \cdots = 2^n q_{n-(n-1)} = 2^{n+1}, \text{ i.e.}$$

(3 – 15) $q_{n+1} = 2^{n+1} (n = 2, 3, \ldots).$

The formula (3 – 15) joint to (3 – 12) yields

(3 – 16) $q_i = 2^i (i = 1, 2, \ldots),$

From (3 – 16) and (3 – 11) we infer

(3 – 17) $l_i = 2^i i! + l_{i-1} \qquad (i = 1, 2, \ldots);$

therefore

$$l_i = 2^i i! + 2^{i-1} (i-1)! + l_{i-2}$$

$$l_i = 2^i i! + 2^{i-1} (i-1)! + l^{i-2} (i-2)! + \cdots + 2^2 2 + 2^1 1! + l_0$$

$$l_i = 1 + \sum_{r=0}^{n} 2^r r! \quad (n = 0, 1, 2, \ldots). \quad \text{Q.E,D.}$$

4. Notations. If T is a tree, then $R_0 T, R_1 T, \ldots$ are its rows or levels.

If $x \in T$, then $T(., x)$ or $(\cdot, x)_T$ denotes the set of all members of T preceding x each. Dually, one defines $T(x, .)$ or $(x, .)_T$.

If r is a relation, then r_1 is its first (or left) part; r_2 is the second part of r.

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