

ON A CLASS OF ARITHMETICAL FUNCTIONS CONNECTED  
 WITH MULTIPLICATIVE FUNCTIONS

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1. Definition and general properties of the functions  $\Lambda_{f,k}(n)$ .

Let  $\Lambda_{f,k}(n)$  be the arithmetical function defined by

$$(1) \quad f(n) \log^k n = \sum_{d|n} f(d) \Lambda_{f,k}\left(\frac{n}{d}\right)$$

where  $f(n)$  is a non-zero arithmetical function,  $k$  is a positive integer and  $\sum_{d|n}$  denotes the summation over all positive divisors of  $n$ . Each function  $\Lambda_{f,k}(n)$  is therefore a uniquely determined arithmetical function depending on the function  $f(n)$  and on the integer  $k$ . In this paper properties of the functions  $\Lambda_{f,k}(n)$  will be investigated, these functions will be explicitly evaluated for most common arithmetical functions, and corresponding asymptotic formulas will be derived. The main reason for introducing the functions  $\Lambda_{f,k}(n)$  is that they represent a natural generalization of two classes of arithmetical functions: they generalize the functions  $\Lambda_k(n)$  investigated in [3] and [4] and the functions  $\Lambda_f(n)$  introduced by B. V. Levin and A. S. Feinleib (see [5] and [6], pp. 379—380).

To see that  $\Lambda_{f,k}(n)$  reduces to  $\Lambda_k(n)$  if  $f(n)=1$  put  $f(n)=1$  in (1) to obtain

$$\log^k n = \sum_{d|n} \Lambda_{1,k}\left(\frac{n}{d}\right) = \sum_{d|n} \Lambda_{1,k}(d)$$

which gives by the Möbius inversion formula

$$\Lambda_{1,k}(n) = \sum_{d|n} \mu(d) \log^k \frac{n}{d} = \Lambda_k(n).$$

On the other hand, if  $k=1$  then (1) gives

$$f(n) \log n = \sum_{d|n} f(d) \Lambda_{f,1}\left(\frac{n}{d}\right),$$

and this is the relation that defines  $\Lambda_f(n)$  so that

$$\Lambda_f(n) = \Lambda_{f,1}(n).$$

Definition (1) can be given in another way: if

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$$

is the Dirichlet series of  $f(n)$  (where it should be assumed that the abscissa of convergence of  $F(s)$  is finite), then term by term differentiation gives

$$F^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} f(n) \log^k n \cdot n^{-s}$$

so that (1) may be written as

$$(2) \quad (-1)^k F^{(k)}(s) = F(s) \sum_{n=1}^{\infty} \Lambda_{f,k}(n) n^{-s}$$

or

$$(3) \quad \frac{(-1)^k F^{(k)}(s)}{F(s)} = \sum_{n=1}^{\infty} \Lambda_{f,k}(n) n^{-s}.$$

**Theorem 1.1.** *Let  $\Lambda_{f,k}(n) = \Lambda_{g,k}(n)$  for two non-zero multiplicative functions  $f(n)$  and  $g(n)$  and for arbitrary  $k$ . Then  $f(n) = g(n)$ .*

**Proof.** Let  $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ ,  $G(s) = \sum_{n=1}^{\infty} g(n) n^{-s}$  and  $\Lambda_{f,k}(n) = \Lambda_{g,k}(n)$ .

Then  $\sum_{n=1}^{\infty} \Lambda_{f,k}(n) n^{-s} = \sum_{n=1}^{\infty} \Lambda_{g,k}(n) n^{-s}$  and so by (3)

$$(4) \quad \frac{(-1)^k F^{(k)}(s)}{F(s)} = \frac{(-1)^k G^{(k)}(s)}{G(s)}$$

and

$$(5) \quad (-1)^k F^{(k)}(s) G(s) = (-1)^k G^{(k)}(s) F(s).$$

By equating coefficients of both sides of (5) we obtain

$$(6) \quad \sum_{d|n} g(d) f\left(\frac{n}{d}\right) \log^k \frac{n}{d} = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) \log^k \frac{n}{d}.$$

We now use induction. Since  $f(n)$  and  $g(n)$  are non-zero multiplicative functions then  $f(1) = g(1) = 1$  and (6) gives when  $n$  is a prime  $p$  that

$$g(1) f(p) \log^k p + g(p) f(1) \log^k 1 = f(1) g(p) \log^k p + f(p) g(1) \log^k 1,$$

that is  $f(p) = g(p)$ . Suppose now that  $f(p^i) = g(p^i)$  for  $i = 1, \dots, a-1$ . If we put  $n = p^a$  in (6) we obtain

$$\begin{aligned} \sum_{i=0}^a g(p^i) f(p^{a-i}) \log^k p^{a-i} &= \sum_{i=0}^a f(p^i) g(p^{a-i}) \log^k p^{a-i}, \\ f(p^a) \log^k p^a + \sum_{i=1}^{a-1} f(p^i) f(p^{a-i}) \log^k p^{a-i} &= \\ \sum_{i=1}^{a-1} f(p^i) f(p^{a-i}) \log^k p^{a-i} + g(p^a) \log^k p^a. \end{aligned}$$

This gives  $f(p^a) = g(p^a)$  for all primes  $p$  and all integers  $a$ , and thus  $f(n) = g(n)$  for all  $n$ , since  $f(n)$  and  $g(n)$  are multiplicative functions.

The preceding theorem tells that for a fixed  $k$  there is a one-to-one correspondence between non-zero multiplicative functions and the functions  $\Lambda_{f,k}(n)$ . Since a great number of arithmetical functions are multiplicative, we may therefore restrict ourselves to the investigation of the function  $\Lambda_{f,k}(n)$  where, if not stated otherwise, from now on  $f(n)$  will be a non-zero multiplicative function.

**Theorem 1.2.** *Let  $k \leq 2$  and  $m < k$ . Then*

$$(7) \quad \Lambda_{f,k}(n) = \Lambda_{f,k-m}(n) \log^m n + \sum_{i=1}^m \binom{m}{i} \sum_{d|n} \Lambda_{f,i}(d) \Lambda_{f,k-m}\left(\frac{n}{d}\right) \log^{m-i} \frac{n}{d}.$$

**Proof.**

$$\begin{aligned} \sum_{n=1}^{\infty} \Lambda_{f,k}(n) n^{-s} &= (-1)^k \frac{F^{(k)}(s)}{F(s)} = \\ &= \frac{(-1)^m}{F(s)} \left[ F(s) \frac{(-1)^{k-m} F^{(k-m)}(s)}{F(s)} \right]^{(m)} = \\ &= \frac{(-1)^m}{F(s)} \left[ F(s) \left( \frac{(-1)^{k-m} F^{(k-m)}(s)}{F(s)} \right)^{(m)} + \sum_{i=1}^m \binom{m}{i} F^{(i)}(s) \left( \frac{(-1)^{k-m} F^{(k-m)}(s)}{F(s)} \right)^{(m-i)} \right] = \\ &= (-1)^m \left[ (-1)^{k-m} \frac{F^{(k-m)}(s)}{F(s)} \right]^{(m)} + \\ &+ \sum_{i=1}^m \left\{ \binom{m}{i} (-1)^i \frac{F^{(i)}(s)}{F(s)} (-1)^{m-i} \left[ (-1)^{k-m} \frac{F^{(k-m)}(s)}{F(s)} \right]^{(m-i)} \right\}. \end{aligned}$$

Using the fact that

$$\begin{aligned} \frac{(-1)^i F^{(i)}(s)}{F(s)} &= \sum_{n=1}^{\infty} \Lambda_{f,i}(n) n^{-s}, \quad (-1)^{k-i} \left[ \frac{F^{(k-m)}(s)}{F(s)} \right]^{(m-i)} = \\ &= \sum_{n=1}^{\infty} \Lambda_{f,k-m}(n) \log^{m-i} n \cdot n^{-s}, \end{aligned}$$

and the uniqueness theorem for Dirichlet series (see [2], pp. 244—245) we obtain (7) after equating coefficients in the last identity.

**Corollary.** *If we set  $m = 1$  in (7) we obtain*

$$(8) \quad \Lambda_{f,k}(n) = \Lambda_{f,k-1}(n) \log n + \sum_{d|n} \Lambda_{f,1}(d) \Lambda_{f,k-1}\left(\frac{n}{d}\right).$$

**Theorem 1.3** *There does not exist a  $k$  for which  $\Lambda_{f,k}(n)$  is multiplicative.*

**Proof.** Let  $(m, n) = 1$ . If  $d_1 | m$ ,  $d_2 | n$  then  $d_1 d_2 | mn$  and conversely, if  $d | mn$  then  $d = d_1 d_2$  where  $d_1 | m$ ,  $d_2 | n$  and  $(d_1, d_2) = 1$ . Suppose  $\Lambda_{f,k}(n)$  multiplicative for some  $k$ . Then

$$\begin{aligned} \Lambda_{f,k}(d_1 d_2) &= \Lambda_{f,k}(d_1) \Lambda_{f,k}(d_2) \\ \sum_{d_1 | m} f\left(\frac{m}{d_1}\right) \sum_{d_2 | n} f\left(\frac{n}{d_2}\right) \Lambda_{f,k}(d_1 d_2) &= \sum_{d_1 | m} f\left(\frac{m}{d_1}\right) \Lambda_{f,k}(d_1) \sum_{d_2 | n} f\left(\frac{n}{d_2}\right) \Lambda_{f,k}(d_2) \\ \sum_{d_1 | m} \sum_{d_2 | n} f\left(\frac{m}{d_1}\right) f\left(\frac{n}{d_2}\right) \Lambda_{f,k}(d_1 d_2) &= f(m) \log^k m \cdot f(n) \log^k n \\ \sum_{d_1 d_2 | mn} f\left(\frac{mn}{d_1 d_2}\right) \Lambda_{f,k}(d_1 d_2) &= f(m) f(n) (\log m \log n)^k \\ \sum_{d | mn} f\left(\frac{mn}{d}\right) \Lambda_{f,k}(d) &= f(m) f(n) (\log m \log n)^k \\ f(mn) \log^k mn &= f(m) f(n) (\log m \log n)^k \end{aligned}$$

Since  $f(n)$  is a non-zero multiplicative function we have

$$\begin{aligned} \log^k mn &= (\log m \log n)^k, \\ \log m + \log n &= \log m \cdot \log n, \end{aligned}$$

which is a contradiction that proves the theorem.

**Theorem 1.4.** Let  $(g, h)$  be a pair of multiplicative functions and  $f(n) = \sum_{d|n} g(d) h\left(\frac{n}{d}\right)$ . Then

$$(9) \quad \Lambda_{f,k}(n) = \Lambda_{g,k}(n) + \Lambda_{h,k}(n) + \sum_{i=1}^{k-1} \binom{k}{i} \sum_{d|n} \Lambda_{g,i}(d) \Lambda_{h,k-i}\left(\frac{n}{d}\right).$$

**Proof.**  $f(n)$  as an arithmetical convolution of two multiplicative functions is also multiplicative. If  $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ ,  $G(s) = \sum_{n=1}^{\infty} g(n) n^{-s}$  and  $H(s) = \sum_{n=1}^{\infty} h(n) n^{-s}$ , then  $f(n) = \sum_{d|n} g(d) h\left(\frac{n}{d}\right)$  gives  $F(s) = G(s) H(s)$  so that

$$\begin{aligned} (-1)^k F^{(k)}(s) &= (-1)^k \left[ G(s) H^{(k)}(s) + H(s) G^{(k)}(s) + \sum_{i=1}^{k-1} \binom{k}{i} G^{(i)}(s) H^{(k-i)}(s) \right] \\ (-1)^k \frac{F^{(k)}(s)}{F(s)} &= (-1)^k \frac{G^{(k)}(s)}{G(s)} + (-1)^k \frac{H^{(k)}(s)}{H(s)} + \\ &\quad \frac{\sum_{i=1}^{k-1} \binom{k}{i} (-1)^i G^{(i)}(s) (-1)^{k-i} H^{(k-i)}(s)}{G(s) H(s)} \end{aligned}$$

Using the uniqueness theorem for Dirichlet series and equating coefficients of  $n^{-s}$  we obtain (9).

**2. Explicit evaluation of  $\Lambda_{f,k}(n)$  for certain arithmetical functions.**

This section gives explicit evaluation of  $\Lambda_{f,k}(n)$  for some most common arithmetical functions  $f(n)$ . The formulas that are obtained involve very often the functions  $\Lambda_k(n)$ , which is to be expected since  $\Lambda_{1,k}(n) = \Lambda_k(n)$  and the functions  $\Lambda_k(n)$  appear also in the following lemma.

**Lemma 2.1.** *If  $g(n)$  is a totally multiplicative function then*

$$(10) \quad \Lambda_{fg,k}(n) = \Lambda_{f,k}(n) g(n).$$

**Proof.** Since  $g(n)$  is a totally multiplicative function  $g(mn) = g(m)g(n)$  for all  $m$  and  $n$  so that

$$\begin{aligned} f(n) \log^k n &= \sum_{d|n} f(d) \Lambda_{f,k}\left(\frac{n}{d}\right), \\ f(n) g(n) \log^k n &= \sum_{d|n} f(d) g\left(d \cdot \frac{n}{d}\right) \Lambda_{f,k}\left(\frac{n}{d}\right), \\ \sum_{d|n} f(d) g(d) \Lambda_{fg,k}\left(\frac{n}{d}\right) &= \sum_{d|n} f(d) g(d) \Lambda_{f,k}\left(\frac{n}{d}\right) g\left(\frac{n}{d}\right). \end{aligned}$$

Therefore  $\Lambda_{fg,k}(n) = \Lambda_{f,k}(n) g(n)$ , and as a corollary we obtain for  $f(n) = 1$  that

$$(11) \quad \Lambda_{g,k}(n) = \Lambda_{1,k}(n) g(n) = \Lambda_k(n) g(n),$$

which gives explicit evaluation of  $\Lambda_{g,k}(n)$  for multiplicative functions  $g(n)$  which are totally multiplicative.

An arithmetical function  $f(n)$  is said to be squarefree if  $f(n) = 0$  whenever there exists  $a > 1$  such that  $a^2 | n$ .

**Lemma 2.2.** *If  $f(n)$  is a squarefree multiplicative function then*

$$(12) \quad \Lambda_{f,k}(n) = \sum_{d|n} \left[ \prod_{p^\alpha || d} (-1)^\alpha f^\alpha(p) \right] f\left(\frac{n}{d}\right) \log^k \frac{n}{d},$$

where  $p^\alpha || d$  means that  $p^\alpha$  divides  $d$  and  $p^{\alpha+1}$  does not.

**Proof.** Since  $f(n)$  is squarefree and multiplicative then

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} f(n) n^{-s} = \prod_p (1 + f(p) p^{-s} + f(p^2) p^{-2s} + \dots) = \prod_p (1 + f(p) p^{-s}). \\ \frac{1}{F(s)} &= \prod_p \frac{1}{1 + f(p) p^{-s}} = \prod_p \sum_{i=0}^{\infty} (-1)^i f^i(p) p^{-is}, \\ \sum_{n=1}^{\infty} \Lambda_{f,k}(n) n^{-s} &= (-1)^k \frac{F^{(k)}(s)}{F(s)} = \\ &= \left[ \sum_{n=1}^{\infty} f(n) \log^k n \cdot n^{-s} \right] \left[ \prod_p \sum_{i=0}^{\infty} (-1)^i f^i(p) p^{-is} \right] \end{aligned}$$

The coefficient of  $n^{-s}$  in the last expression equals

$$\sum_{d|n} \left[ \prod_{p^\alpha || d} (-1)^\alpha f^\alpha(p) \right] f\left(\frac{n}{d}\right) \log^k \frac{n}{d},$$

so that (12) is proved.

*Evaluation of  $\Lambda_{\mu,k}(n)$ .* The Möbius function  $\mu(n)$  is squarefree and thus lemma 2.2 may be used. Moreover,  $\mu(p) = -1$ ,  $\prod_{p^\alpha || d} (-1)^\alpha f^\alpha(p) = 1$  and therefore

$$(13) \quad \Lambda_{\mu,k}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log^k \frac{n}{d} = \sum_{d|n} \mu(d) \log^k d.$$

Since  $\Lambda_k(n) = \sum_{d|n} \mu(d) \log^k \frac{n}{d}$ , which is similar to (13), it might be expected that  $\Lambda_{\mu,k}(n)$  could be expressed in some way by the functions  $\Lambda_k(n)$ . This is indeed so as

$$\begin{aligned} \Lambda_{\mu,k}(n) &= \sum_{d|n} \mu(d) \log^k d = \sum_{d|n} \mu(d) \left( \log n - \log \frac{n}{d} \right)^k = \\ &= \sum_{d|n} \mu(d) \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \log^i n \cdot \log^{k-i} \frac{n}{d} = \\ &= \sum_{d|n} \mu(d) \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k}{i} \log^i n \cdot \log^{k-i} \frac{n}{d} + \sum_{d|n} \mu(d) \log^k n = \\ &= \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k}{i} \log^i n \sum_{d|n} \mu(d) \log^{k-i} \frac{n}{d} = \\ &= \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k}{i} \Lambda_{k-i}(n) \log^i n. \end{aligned}$$

This gives for  $k = 1$

$$(14) \quad \Lambda_{\mu,1}(n) = \Lambda_\mu(n) = -\Lambda_1(n) = -\Lambda(n),$$

where  $\Lambda(n)$  is the von Mangoldt function.

*Evaluation of  $\Lambda_{\chi,k}(n)$ .* Since all characters  $\chi(n)$  for a given modulus  $i$  are totally multiplicative functions, (11) gives immediately

$$(15) \quad \Lambda_{\chi,k}(n) = \chi(n) \Lambda_k(n).$$

*Evaluation of  $\Lambda_{\tau,k}(n)$ .*  $\tau(n) = \sum_{d|n} 1$  is the number of divisors function. If we use theorem 1.4 with  $g(n) = h(n) = 1$  then

$$(16) \quad \Lambda_{\tau,k}(n) = 2\Lambda_k(n) + \sum_{i=1}^{k-1} \binom{k}{i} \sum_{d|n} \Lambda_i(d) \Lambda_{k-i}\left(\frac{n}{d}\right).$$

*Evaluation of  $\Lambda_{\sigma_i,k}(n)$ .*  $\sigma_i(n) = \sum_{d|n} d^i = \sum_{d|n} \left(\frac{n}{d}\right)^i$  is the sum of divisor powers function, where  $i > 0$ . Theorem 1.4 can be used with  $g(n) = 1$ ,  $h(n) = n^i$ ,  $\Lambda_{h,k}(n) = n^i \Lambda_k(n)$  (since  $h(n)$  is totally multiplicative) to give

$$(17) \quad \Lambda_{\sigma_i,k}(n) = \Lambda_k(n) (1 + n^i) + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{d|n} \Lambda_{k-j}(d) \Lambda_j\left(\frac{n}{d}\right) \left(\frac{n}{d}\right)^i.$$

*Evaluation of  $\Lambda_{\phi,k}(n)$ .* Euler's totient function  $\phi(n)$  can be expressed as  $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$ , and therefore theorem 1.4 may be used with  $g(n) = \mu(n)$ ,  $h(n) = n$  to obtain

$$(18) \quad \Lambda_{\phi,k}(n) = \Lambda_{\mu,k}(n) + n\Lambda_k(n) + \sum_{i=1}^{k-1} \binom{k}{i} \sum_{d|n} \Lambda_{\mu,i}(d) \Lambda_{k-i}\left(\frac{n}{d}\right) \frac{n}{d}.$$

*Evaluation of  $\Lambda_{r,k}(n)$ .* The function  $r(n)$  is the number of representations of  $n$  as a sum of two squares. It is known ([2], pp. 241—242) that  $r(n) = 4 \sum_{d|n} \chi(d)$  where

$$\chi(n) = \begin{cases} (-1)^{\frac{n(n-1)}{2}} & n = 2k + 1 \\ 0 & n = 2k \end{cases}$$

and the constant 4 comes from all possible combinations of signs when  $n = x^2 + y^2$ . The function  $\chi(n)$  is the non-principal character mod 4 and therefore  $\Lambda_{\chi,k}(n) = \chi(n) \Lambda_k(n)$ . If we put  $r(n) = 4s(n)$  where  $s(n) = \sum_{d|n} \chi(d)$  then

$$\begin{aligned} \sum_{n=1}^{\infty} \Lambda_{r,k}(n) n^{-s} &= (-1)^k \frac{R^{(k)}(s)}{R(s)} = (-1)^k \frac{\left(\sum_{n=1}^{\infty} r(n) n^{-s}\right)^{(k)}}{\sum_{n=1}^{\infty} r(n) n^{-s}} = \\ &= (-1)^k \frac{\left(\sum_{n=1}^{\infty} 4s(n) n^{-s}\right)^{(k)}}{\sum_{n=1}^{\infty} 4s(n) n^{-s}} = (-1)^k \frac{S^{(k)}(s)}{S(s)} = \sum_{n=1}^{\infty} \Lambda_{s,k}(n) n^{-s}, \end{aligned}$$

so that  $\Lambda_{r,k}(n) = \Lambda_{s,k}(n)$ . For  $s(n)$  theorem 1.4 may be used with  $g(n) = 1$ ,  $h(n) = \chi(n)$  and therefore

$$(19) \quad \Lambda_{r,k}(n) = \Lambda_{s,k}(n) = (1 + \chi(n)) \Lambda_k(n) + \sum_{i=1}^{\infty} \binom{k}{i} \sum_{d|n} \Lambda_i(d) \Lambda_{k-i}\left(\frac{n}{d}\right) \chi\left(\frac{n}{d}\right).$$

3. Asymptotic formulas.

This section contains asymptotic formulas for  $\sum_{n \leq x} \Lambda_{f,k}(n)$  when  $f(n)$  is  $\tau(n)$ ,  $\mu(n)$  or  $\phi(n)$ , and a general theorem about  $\sum_{n \leq x} \Lambda_{f,k}(n)$  with applications to functions connected with the two-square problem and to the function  $\tau_m(n)$  which is the number of representations of  $n$  as a product of  $m$  factors.

**Theorem 3.1**  $\sum_{n \leq x} \Lambda_{\tau,k}(n) = k(k+1)x \log^{k-1} x + O(x \log^{k-1} x).$

**Proof.** Using (16) and formulas

$$(20) \quad \sum_{n \leq x} \Lambda_k(n) = kx \log^{k-1} x + O(x \log^{k-2} x),$$

$$(21) \quad \sum_{mn \leq x} \Lambda_r(m) \Lambda_s(n) = \frac{r! s!}{(r+s-1)!} x \log^{r+s-1} x + O(x \log^{r+s-2} x)$$

proved in [3] we have

$$\begin{aligned} \sum_{n \leq x} \Lambda_{\tau,k}(n) &= \sum_{n \leq x} 2 \Lambda_k(n) + \sum_{i=1}^{k-1} \binom{k}{i} \sum_{n \leq x} \sum_{d|n} \Lambda_i(d) \Lambda_{k-i}\left(\frac{n}{d}\right) = \\ &2kx \log^{k-1} x + O(x \log^{k-2} x) + \sum_{i=1}^{k-1} \binom{k}{i} \sum_{mn \leq x} \Lambda_i(m) \Lambda_{k-i}(n) = \\ &\sum_{i=1}^{k-1} \frac{k! i! (k-i)! x \log^{k-1} x}{(k-i)! i! (k-1)!} + 2kx \log^{k-1} x + O(x \log^{k-2} x) = \\ &2kx \log^{k-1} x + \sum_{i=1}^{k-1} kx \log^{k-1} x + O(x \log^{k-2} x) = \\ &k(k+1)x \log^{k-1} x + O(x \log^{k-2} x). \end{aligned}$$

**Theorem 3.2.**  $\sum_{n \leq x} \Lambda_{\mu,k}(n) = \begin{cases} -x + O\left(\frac{x}{\log x}\right) & k = 1 \\ O(x \log^{k-2} x) & k \geq 2 \end{cases}.$

**Proof.** Partial summation of (20) gives

$$(22) \quad \sum_{n \leq x} \Lambda_k(n) \log^i n = kx \log^{k+i-1} x + O(x \log^{k+i-2} x).$$





Partial summation of (20) gives

$$\sum_{n \leq x} n \Lambda_k(n) = \frac{kx^2}{2} \log^{k-1} x + O(x^2 \log^{k-2} x),$$

which proves the theorem for  $k \geq 2$ , but since  $\Lambda_{\phi,1}(n) = (n-1) \Lambda_1(n)$  it is easily seen that the theorem is also true when  $k = 1$ .

Before passing on to theorem 3.4, it is necessary to prove the following lemma.

**Lemma 3.1** *Let  $f(n)$  be a non-negative (not necessarily multiplicative) arithmetical function and let for  $m \geq 2$*

$$\sum_{n \leq x} f(n) = A \log^m x + O(\log^{m-1} x),$$

where  $A \neq 0$ . Then

$$\sum_{n \leq x} \frac{f(n)}{\log \frac{2x}{n}} = O(\log^{m-1} x).$$

**Proof.** Since by hypothesis we have

$$\sum_{n \leq x} \frac{f(n)}{\log x} = A \log^{m-1} x + O(\log^{m-2} x) = O(\log^{m-1} x),$$

it is sufficient to show that

$$\begin{aligned} \sum_{n \leq x} f(n) \left( \frac{1}{\log \frac{2x}{n}} - \frac{1}{\log x} \right) &= O(\log^{m-1} x). \\ \sum_{n \leq x} f(n) \left( \frac{1}{\log \frac{2x}{n}} - \frac{1}{\log x} \right) &= \sum_{n \leq x} f(n) \int_{2x/n}^x \frac{dt}{t \log^2 t} = \\ &= O(1) + \int_2^x \frac{\sum_{2x/t \leq n \leq x} f(n)}{t \log^2 t} dt = \\ &= O(1) + \int_2^x \frac{A \log^m x + O(\log^{m-1} x) - A \log^m 2x/t + O(\log^{m-1} 2x/t)}{t \log^2 t} dt = \\ &= O(1) + \int_2^x \frac{O(\log^{m-1} x) + C \log x \cdot \log^{m-1} t + O(\log^{m-1} x)}{t \log^2 t} dt = O(\log^{m-1} x). \end{aligned}$$

**Theorem 3.4** *Suppose  $\Lambda_{f,1}(n) \geq 0$  and*

$$(23) \quad \sum_{n \leq x} \Lambda_{f,1}(n) = Ax + O\left(\frac{x}{\log x}\right),$$

where  $A \neq 0$ . Then

$$(24) \quad \sum_{n \leq x} \Lambda_{f,k}(n) = A \prod_{i=1}^{k-1} \left(1 + \frac{A}{i}\right) \cdot x \log^{k-1} x + O(x \log^{k-2} x).$$

**Proof.** Induction on  $k$ . Since (24) reduces to (23) when  $k=1$ , the theorem is true for  $k=1$ . (23) in fact generalizes the prime number theorem, since for  $A=1, f(n)=1$  we obtain

$$\sum_{n \leq x} \Lambda_{1,1}(n) = \sum_{n \leq x} \Lambda(n) = x + O\left(\frac{x}{\log x}\right),$$

which is a version of the prime number theorem (see chs. I and III of [1]) with a weak error term. Thus the hypothesis made in (23) is a natural one, since many arithmetical functions have asymptotic distributions similar to the distribution of primes.

Suppose now the theorem is true for some  $k$ , then by (8)

$$\Lambda_{f,k+1}(n) = \Lambda_{f,k}(n) \log n + \sum_{d|n} \Lambda_{f,k}(d) \Lambda_{f,1}\left(\frac{n}{d}\right),$$

which shows that  $\Lambda_{f,1}(n) \geq 0$  implies  $\Lambda_{f,k}(n) \geq 0$  for all  $k$ . Summation on  $n$  gives

$$\begin{aligned} \sum_{n \leq x} \Lambda_{f,k+1}(n) &= \sum_{n \leq x} \Lambda_{f,k}(n) \log n + \sum_{n \leq x} \sum_{d|n} \Lambda_{f,k}(d) \Lambda_{f,1}\left(\frac{n}{d}\right) = \\ &= \sum_{n \leq x} \Lambda_{f,k}(n) \log n + \sum_{n \leq x} \Lambda_{f,k}(n) \sum_{m \leq x/n} \Lambda_{f,1}(m). \end{aligned}$$

Since for  $x \geq 2$   $\frac{1}{\log x} \leq \frac{2}{\log 2x}$ , (23) may be written as

$$(25) \quad \sum_{n \leq x} \Lambda_{f,1}(n) = Ax + O\left(\frac{x}{\log 2x}\right),$$

which is easier to work with because  $\frac{1}{\log \frac{2x}{n}}$  stays bounded for  $n=x$ , while  $\frac{1}{\log \frac{x}{n}}$  does not.

Partial summation and induction hypothesis give

$$(26) \quad \sum_{n \leq x} \Lambda_{f,k}(n) \log n = A \prod_{i=1}^{k-1} \left(1 + \frac{A}{i}\right) \cdot x \log^k x + O(x \log^{k-1} x),$$

$$(27) \quad \sum_{n \leq x} \frac{\Lambda_{f,k}(n)}{n} = \frac{A}{k} \prod_{i=1}^{k-1} \left(1 + \frac{A}{i}\right) \log^k x + O(\log^{k-1} x).$$

Therefore

$$\begin{aligned} & \sum_{n \leq x} \Lambda_{f,k}(n) \sum_{m \leq x/n} \Lambda_{f,1}(m) = \sum_{n \leq x} \Lambda_{f,k}(n) \left[ \frac{Ax}{n} + O\left(\frac{x}{n \log \frac{2x}{n}}\right) \right] = \\ & \frac{A^2}{k} \cdot \prod_{i=1}^{k-1} \left(1 + \frac{A}{i}\right) \cdot x \log^k x + O(x \log^{k-1} x) + O\left(x \sum_{n \leq x} \frac{\Lambda_{f,k}(n)}{n \log \frac{2x}{n}}\right) = \\ (28) \quad & \frac{A^2}{k} \cdot \prod_{i=1}^{k-1} \left(1 + \frac{A}{i}\right) \cdot x \log^k x + O(x \log^{k-1} x), \end{aligned}$$

where lemma 3.1 was used with  $f(n) = \frac{1}{n} \Lambda_{f,k}(n)$ ,  $m = k$ . Addition of (26) and (28) gives

$$\begin{aligned} & \sum_{n \leq x} \Lambda_{f,k+1}(n) = \\ & A \prod_{i=1}^{k-1} \left(1 + \frac{A}{i}\right) \cdot x \log^k x + \frac{A^2}{k} \cdot \prod_{i=1}^{k-1} \left(1 + \frac{A}{i}\right) \cdot x \log^k x + O(x \log^{k-1} x) = \\ & A \prod_{i=1}^k \left(1 + \frac{A}{i}\right) \cdot x \log^k x + O(x \log^{k-1} x), \end{aligned}$$

which ends the proof of theorem 3.4.

Consider first the application of theorem 3.4 to the function  $b(n)$ , the characteristic function of numbers that are a sum of two squares:

$$b(n) = \begin{cases} 1 & n = x^2 + y^2 \\ 0 & n \neq x^2 + y^2 \end{cases}.$$

The function  $b(n)$  is multiplicative, and from the formula  $r(n) = 4 \sum_{d|n} \chi(d)$  it is easily seen that the multiplicative semigroup of numbers that are a sum of two squares is generated by 2, by primes of the form  $4k + 1$  and by squares of the primes of the form  $4k + 3$ . Therefore

$$B(s) = \sum_{n=1}^{\infty} b(n) n^{-s} = (1 - 2^{-s})^{-1} \prod_{p \equiv 1 \pmod{4}} (1 - p^{-s})^{-1} \prod_{p \equiv 3 \pmod{4}} (1 - p^{-2s})^{-1}.$$

$$\log B(s) = -\log(1 - 2^{-s}) - \log \prod_{p \equiv 1 \pmod{4}} (1 - p^{-s}) - \log \prod_{p \equiv 3 \pmod{4}} (1 - p^{-2s}),$$

$$\sum_{n=1}^{\infty} \Lambda_{b,1}(n) n^{-s} = -\frac{B'(s)}{B(s)} = -[\log B(s)]' =$$

$$\sum_{i=0}^{\infty} \frac{\log 2}{2^{is}} + \sum_{p \equiv 1 \pmod{4}} \sum_{i=0}^{\infty} \frac{\log p}{p^{is}} + \sum_{p \equiv 3 \pmod{4}} \sum_{i=0}^{\infty} \frac{2 \log p}{p^{2is}},$$

so that

$$\Lambda_{b,1}(n) = \begin{cases} \log p & n = p^i \quad p = 2, \quad p \equiv 1 \pmod{4} \\ 2 \log p & n = p^{2i} \quad p \equiv 3 \pmod{4} \end{cases}$$

$$\sum_{n \leq x} \Lambda_{b,1}(n) = \sum_{2^i \leq x} \log 2 + \sum_{\substack{p^i \leq x \\ p \equiv 1 \pmod{4}}} \log p + \sum_{\substack{p^{2i} \leq x \\ p \equiv 3 \pmod{4}}} \log p =$$

$$\frac{x}{2} + O\left(\frac{x}{\log x}\right),$$

where the theorem for prime numbers in arithmetical progressions ([7], p. 157) and partial summation are used.

Theorem 3.4 may now be applied with  $A = \frac{1}{2}$  so that

$$(29) \quad \sum_{n \leq x} \Lambda_{b,k}(n) = \frac{1}{2} \prod_{i=1}^{k-1} \left(1 + \frac{1}{2^i}\right) x \log^{k-1} x + O(x \log^{k-2} x) =$$

$$\frac{(2k-1)!!}{2^k (k-1)!} x \log^{k-1} x + O(x \log^{k-2} x).$$

To obtain the asymptotic formula for  $\sum_{n \leq x} \Lambda_{r,k}(n)$  note that by (19)

$$\Lambda_{r,1}(n) = \Lambda(n) + \chi(n) \Lambda(n)$$

so that  $\Lambda_{r,1}(n) \geq 0$  since

$$\chi(n) \Lambda(n) = \begin{cases} 0 & n \neq p^i \\ 0 & n = 2^i \\ \log p & n = p^i \quad p \equiv 1 \pmod{4} \\ (-1)^i \log p & n = p^i \quad p \equiv 3 \pmod{4} \end{cases}$$

$$\sum_{n \leq x} \Lambda_{r,1}(n) = \sum_{n \leq x} \Lambda(n) + \sum_{n \leq x} \chi(n) \Lambda(n) =$$

$$x + O\left(\frac{x}{\log x}\right) + \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{4}}} \log p + \sum_{\substack{p \leq x \\ p \equiv 3 \pmod{4}}} (-\log p) =$$

$$x + O\left(\frac{x}{\log x}\right) + \left(\frac{x}{2} + O\left(\frac{x}{\log x}\right)\right) + \left(-\frac{x}{2} + O\left(\frac{x}{\log x}\right)\right) =$$

$$x + O\left(\frac{x}{\log x}\right),$$

where the prime number theorem for primes in an arithmetical progression was used again. Theorem 3.4 gives then ( $A = 1$ )

$$(30) \quad \sum_{n \leq x} \Lambda_{r,k}(n) = kx \log^{k-1} x + O(x \log^{k-2} x).$$

Finally, let  $\tau_m(n) = \sum_{a_1 a_2 \cdots a_m = n} 1$  be the number of representations of  $n$  as a product of  $m$  factors (of which any may be unity). Since ([2], p. 255)  $\zeta^m(s) = \sum_{n=1}^{\infty} \tau_m(n) n^{-s}$  we have

$$\sum_{n=1}^{\infty} \Lambda_{\tau_{m,1}}(n) n^{-s} = [-\log \zeta^m(s)]' = -m \frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} m \Lambda(n) n^{-s}$$

so that

$$\Lambda_{\tau_{m,1}}(n) = m \Lambda(n).$$

The hypotheses of theorem 3.4 are satisfied since  $\Lambda(n) \geq 0$ ,

$$\sum_{n \leq x} \Lambda_{\tau_{m,1}}(n) = m \sum_{n \leq x} \Lambda(n) = mx + O\left(\frac{x}{\log x}\right),$$

so that we obtain with  $A = m$

**Theorem 3.5.**  $\sum_{n \leq x} \Lambda_{\tau_{m,k}}(n) = m \prod_{i=1}^{k-1} \frac{m+i}{i} \cdot x \log^{k-1} x + O(x \log^{k-2} x).$

If we set  $m = 2$  we get another proof of theorem 3.1 since

$$2 \prod_{i=1}^{k-1} \frac{2+i}{i} = k(k+1), \quad \text{and} \quad \tau_2(n) = \sum_{a_1 a_2 = n} 1 = \sum_{d|n} 1 = \tau(n).$$

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