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ON A CLASS OF ARITHMETICAL FUNCTIONS CONNECTED WITH MULTIPLICATIVE FUNCTIONS

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1. Definition and general properties of the functions $\Lambda_{f,k}(n)$.

Let $\Lambda_{f,k}(n)$ be the arithmetical function defined by

$$f(n)\log^k n=\sum_{d|n} f(d)\,\Lambda_{f,k}\left(\frac{n}{d}\right) \tag{1}$$

where $f(n)$ is a non-zero arithmetical function, k is a positive integer and $\sum_{d|n}$ denotes the summation over all positive divisors of n. Each function $\Lambda_{f,k}(n)$ is therefore a uniquely determined arithmetical function depending on the function $f(n)$ and on the integer k. In this paper properties of the functions $\Lambda_{f,k}(n)$ will be investigated, these functions will be explicitly evaluated for most common arithmetical functions, and corresponding asymptotic formulas will be derived. The main reason for introducing the functions $\Lambda_{f,k}(n)$ is that they represent a natural generalization of two classes of arithmetical functions: they generalize the functions $\Lambda_k(n)$ investigated in [3] and [4] and the functions $\Lambda_f(n)$ introduced by B. V. Levin and A. S. Feinleib (see [5] and [6], pp. 379—380).

To see that $\Lambda_{f,k}(n)$ reduces to $\Lambda_k(n)$ if $f(n)=1$ put $f(n)=1$ in (1) to obtain

$$\log^k n=\sum_{d|n}\Lambda_{1,k}\left(\frac{n}{d}\right)=\sum_{d|n}\Lambda_{1,k}(d)$$

which gives by the Möbius inversion formula

$$\Lambda_{1,k}(n)=\sum_{d|n}\mu(d)\log^k\frac{n}{d}=\Lambda_k(n).$$

On the other hand, if $k=1$ then (1) gives

$$f(n)\log n=\sum_{d|n} f(d)\,\Lambda_{f,1}\left(\frac{n}{d}\right),$$

and this is the relation that defines $\Lambda_f(n)$ so that

$$\Lambda_f(n)=\Lambda_{f,1}(n).$$

Definition (1) can be given in another way: if

$$F(s)=\sum_{n=1}^{\infty} f(n)\, n^{-s}$$

is the Dirichlet series of $f(n)$ (where it should be assumed that the abscissa of convergence of $F(s)$ is finite), then term by term differentiation gives

$$F^{(k)}(s)=(-1)^k \sum_{n=1}^{\infty} f(n) \log^k n \cdot n^{-s}$$

so that (1) may be written as

$$(-1)^k F^{(k)}(s)=F(s) \sum_{n=1}^{\infty} \Lambda_{f,k}(n)\, n^{-s} \tag{2}$$

or

$$\frac{(-1)^k F^{(k)}(s)}{F(s)}=\sum_{n=1}^{\infty} \Lambda_{f,k}(n)\, n^{-s}. \tag{3}$$

Theorem 1.1. *Let $\Lambda_{f,k}(n)=\Lambda_{g,k}(n)$ for two non-zero multiplicative functions $f(n)$ and $g(n)$ and for arbitrary k. Then $f(n)=g(n)$.*

Proof. Let $F(s)=\sum_{n=1}^{\infty} f(n)\, n^{-s}$, $G(s)=\sum_{n=1}^{\infty} g(n) n^{-s}$ and $\Lambda_{f,k}(n)=\Lambda_{g,k}(n)$. Then $\sum_{n=1}^{\infty} \Lambda_{f,k}(n) n^{-s}=\sum_{n=1}^{\infty} \Lambda_{g,k}(n)\, n^{-s}$ and so by (3)

$$\frac{(-1)^k F^{(k)}(s)}{F(s)}=\frac{(-1)^k G^{(k)}(s)}{G(s)} \tag{4}$$

and

$$(-1)^k F^{(k)}(s)\, G(s)=(-1)^k G^{(k)} F(s). \tag{5}$$

By equating coefficients of both sides of (5) we obtain

$$\sum_{d|n} g(d) f\left(\frac{n}{d}\right) \log^k \frac{n}{d}=\sum_{d|n} f(d)\, g\left(\frac{n}{d}\right) \log^k \frac{n}{d}. \tag{6}$$

We now use induction. Since $f(n)$ and $g(n)$ are non-zero multiplicative functions then $f(1)=g(1)=1$ and (6) gives when n is a prime p that

$$g(1) f(p) \log^k p+g(p) f(1) \log^k 1=f(1)\, g(p) \log^k p+f(p)\, g(1) \log^k 1,$$

that is $f(p)=g(p)$. Suppose now that $f(p^i)=g(p^i)$ for $i=1, \ldots, a-1$. If we put $n=p^a$ in (6) we obtain

$$\sum_{i=0}^{a} g(p^i) f(p^{a-i}) \log^k p^{a-i}=\sum_{i=0}^{a} f(p^i)\, g(p^{a-i}) \log^k p^{a-i},$$

$$f(p^a) \log^k p^a+\sum_{i=1}^{a-1} f(p^i) f(p^{a-i}) \log^k p^{a-i}=$$

$$\sum_{i=1}^{a-1} f(p^i) f(p^{a-i}) \log^k p^{a-i}+g(p^a) \log^k p^a.$$

This gives $f(p^a)=g(p^a)$ for all primes p and all integers a, and thus $f(n)=g(n)$ for all n, since $f(n)$ and $g(n)$ are multiplicative functions.

The preceding theorem tells that for a fixed k there is a one-to-one correspondence between non-zero multiplicative functions and the functions $\Lambda_{f,k}(n)$. Since a great number of arithmetical functions are multiplicative, we may therefore restrict ourselves to the investigation of the function $\Lambda_{f,k}(n)$ where, if not stated otherwise, from now on $f(n)$ will be a non-zero multiplicative function.

Theorem 1.2. *Let $k\leq 2$ and $m<k$. Then*

$$\Lambda_{f,k}(n)=\Lambda_{f,k-m}(n)\log^m n+\sum_{i=1}^{m}\binom{m}{i}\sum_{d|n}\Lambda_{f,i}(d)\,\Lambda_{f,k-m}\left(\frac{n}{d}\right)\log^{m-i}\frac{n}{d}. \tag{7}$$

Proof. $\sum_{n=1}^{\infty}\Lambda_{f,k}(n)\,n^{-s}=(-1)^k\dfrac{F^{(k)}(s)}{F(s)}=$

$$\frac{(-1)^m}{F(s)}\left[F(s)\,\frac{(-1)^{k-m}F^{(k-m)}(s)}{F(s)}\right]^{(m)}=$$

$$\frac{(-1)^m}{F(s)}\left[F(s)\left(\frac{(-1)^{k-m}F^{(k-m)}(s)}{F(s)}\right)^{(m)}+\sum_{i=1}^{m}\binom{m}{i}F^{(i)}(s)\left(\frac{(-1)^{k-m}F^{(k-m)}(s)}{F(s)}\right)^{(m-i)}\right]=$$

$$(-1)^m\left[(-1)^{k-m}\frac{F^{(k-m)}(s)}{F(s)}\right]^{(m)}+$$

$$+\sum_{i=1}^{m}\left\{\binom{m}{i}(-1)^i\frac{F^{(i)}(s)}{F(s)}(-1)^{m-i}\left[(-1)^{k-m}\frac{F^{(k-m)}(s)}{F(s)}\right]^{(m-i)}\right\}.$$

Using the fact that

$$\frac{(-1)^iF^{(i)}(s)}{F(s)}=\sum_{n=1}^{\infty}\Lambda_{f,i}(n)\,n^{-s},\quad (-1)^{k-i}\left[\frac{F^{(k-m)}(s)}{F(s)}\right]^{(m-i)}=$$

$$\sum_{n=1}^{\infty}\Lambda_{f,k-m}(n)\log^{m-i}n\cdot n^{-s},$$

and the uniqueness theorem for Dirichlet series (see [2], pp. 244—245) we obtain (7) after equating coefficients in the last identity.

Corollary. *If we set $m=1$ in (7) we obtain*

$$\Lambda_{f,k}(n)=\Lambda_{f,k-1}(n)\log n+\sum_{d|n}\Lambda_{f,1}(d)\,\Lambda_{f,k-1}\left(\frac{n}{d}\right). \tag{8}$$

Theorem 1.3 *There does not exist a k for which $\Lambda_{f,k}(n)$ is multiplicative.*

Proof. Let $(m, n)=1$. If $d_1|m$, $d_2|n$ then $d_1 d_2|mn$ and conversely, if $d|mn$ then $d=d_1 d_2$ where $d_1|m$, $d_2|n$ and $(d_1, d_2)=1$. Suppose $\Lambda_{f,k}(n)$ multiplicative for some k. Then

$$\Lambda_{f,k}(d_1 d_2)=\Lambda_{f,k}(d_1)\Lambda_{f,k}(d_2)$$

$$\sum_{d_1|m} f\left(\frac{m}{d_1}\right)\sum_{d_2|n} f\left(\frac{n}{d_2}\right)\Lambda_{f,k}(d_1 d_2)=\sum_{d_1|m} f\left(\frac{m}{d_1}\right)\Lambda_{f,k}(d_1)\sum_{d_2|n} f\left(\frac{n}{d_2}\right)\Lambda_{f,k}(d_2)$$

$$\sum_{d_1|m}\sum_{d_2|n} f\left(\frac{m}{d_1}\right) f\left(\frac{n}{d_2}\right)\Lambda_{f,k}(d_1 d_2)=f(m)\log^k m\cdot f(n)\log^k n$$

$$\sum_{d_1 d_2|mn} f\left(\frac{mn}{d_1 d_2}\right)\Lambda_{f,k}(d_1 d_2)=f(m)f(n)\,(\log m\log n)^k$$

$$\sum_{d|mn} f\left(\frac{mn}{d}\right)\Lambda_{f,k}(d)=f(m)f(n)\,(\log m\log n)^k$$

$$f(mn)\log^k mn=f(m)f(n)\,(\log m\log n)^k$$

Since $f(n)$ is a non-zero multiplicative function we have

$$\log^k mn=(\log m\log n)^k,$$

$$\log m+\log n=\log m\cdot\log n,$$

which is a contradiction that proves the theorem.

Theorem 1.4. *Let (g, h) be a pair of multiplicative functions and $f(n)=\sum_{d|n} g(d)\,h\left(\frac{n}{d}\right)$. Then*

$$\Lambda_{f,k}(n)=\Lambda_{g,k}(n)+\Lambda_{h,k}(n)+\sum_{i=1}^{k-1}\binom{k}{i}\sum_{d|n}\Lambda_{g,i}(d)\,\Lambda_{h,k-i}\left(\frac{n}{d}\right). \tag{9}$$

Proof. $f(n)$ as an arithmetical convolution of two multiplicative functions is also multiplicative. If $F(s)=\sum_{n=1}^{\infty} f(n)n^{-s}$, $G(s)=\sum_{n=1}^{\infty} g(n)n^{-s}$ and $H(s)=\sum_{n=1}^{\infty} h(n)n^{-s}$, then $f(n)=\sum_{d|n} g(d)\,h\left(\frac{n}{d}\right)$ gives $F(s)=G(s)\,H(s)$ so that

$$(-1)^k F^{(k)}(s)=(-1)^k\left[G(s)\,H^{(k)}(s)+H(s)\,G^{(k)}(s)+\sum_{i=1}^{k-1}\binom{k}{i}G^{(i)}(s)\,H^{(k-i)}(s)\right]$$

$$(-1)^k\frac{F^{(k)}(s)}{F(s)}=(-1)^k\frac{G^{(k)}(s)}{G(s)}+(-1)^k\frac{H^{(k)}(s)}{H(s)}+$$

$$\sum_{i=1}^{k-1}\binom{k}{i}\frac{(-1)^i G^{(i)}(s)\,(-1)^{k-i}H^{(k-i)}(s)}{G(s)\,H(s)}$$

Using the uniqueness theorem for Dirichlet series and equating coefficients of n^{-s} we obtain (9).

2. Explicit evaluation of $\Lambda_{f,k}(n)$ for certain arithmetical functions.

This section gives explicit evaluation of $\Lambda_{f,k}(n)$ for some most common arithmetical functions $f(n)$. The formulas that are obtained involve very often the functions $\Lambda_k(n)$, which is to be expected since $\Lambda_{1,k}(n)=\Lambda_k(n)$ and the functions $\Lambda_k(n)$ appear also in the following lemma.

Lemma 2.1. *If $g(n)$ is a totally multiplicative function then*

$$\Lambda_{fg,k}(n)=\Lambda_{f,k}(n)\,g(n). \tag{10}$$

Proof. Since $g(n)$ is a totally multiplicative function $g(mn)=g(m)\,g(n)$ for all m and n so that

$$f(n)\log^k n=\sum_{d\mid n} f(d)\,\Lambda_{f,k}\left(\frac{n}{d}\right),$$

$$f(n)\,g(n)\log^k n=\sum_{d\mid n} f(d)\,g\left(d\cdot\frac{n}{d}\right)\Lambda_{f,k}\left(\frac{n}{d}\right),$$

$$\sum_{d\mid n} f(d)\,g(d)\,\Lambda_{fg,k}\left(\frac{n}{d}\right)=\sum_{d\mid n} f(d)\,g(d)\,\Lambda_{f,k}\left(\frac{n}{d}\right)g\left(\frac{n}{d}\right).$$

Therefore $\Lambda_{fg,k}(n)=\Lambda_{f,k}(n)\,g(n)$, and as a corollary we obtain for $f(n)=1$ that

$$\Lambda_{g,k}(n)=\Lambda_{1,k}(n)\,g(n)=\Lambda_k(n)\,g(n), \tag{11}$$

which gives explicit evaluation of $\Lambda_{g,k}(n)$ for multiplicative functions $g(n)$ which are totally multiplicative.

An arithmetical function $f(n)$ is said to be squarefree if $f(n)=0$ whenever there exists $a>1$ such that $a^2\mid n$.

Lemma 2.2. *If $f(n)$ is a squarefree multiplicative function then*

$$\Lambda_{f,k}(n)=\sum_{d\mid n}\left[\prod_{p^\alpha\|d}(-1)^\alpha f^\alpha(p)\right]f\left(\frac{n}{d}\right)\log^k\frac{n}{d}, \tag{12}$$

where $p^\alpha\|d$ means that p^α divides d and $p^{\alpha+1}$ does not.

Proof. Since $f(n)$ is squarefree and multiplicative then

$$F(s)=\sum_{n=1}^{\infty} f(n)\,n^{-s}=\prod_p\left(1+f(p)\,p^{-s}+f(p^2)\,p^{-2s}+\cdots\right)=\prod_p\left(1+f(p)\,p^{-s}\right).$$

$$\frac{1}{F(s)}=\prod_p\frac{1}{1+f(p)\,p^{-s}}=\prod_p\sum_{i=0}^{\infty}(-1)^i f^i(p)\,p^{-is},$$

$$\sum_{n=1}^{\infty}\Lambda_{f,k}(n)\,n^{-s}=(-1)^k\frac{F^{(k)}(s)}{F(s)}=$$

$$\left[\sum_{n=1}^{\infty} f(n)\log^k n\cdot n^{-s}\right]\left[\prod_p\sum_{i=0}^{\infty}(-1)^i f^i(p)\,p^{-is}\right]$$

The coefficient of n^{-s} in the last expression equals

$$\sum_{d|n}\left[\prod_{p^\alpha \| d}(-1)^\alpha f^\alpha(p)\right] f\left(\frac{n}{d}\right)\log^k\frac{n}{d},$$

so that (12) is proved.

Evaluation of $\Lambda_{\mu,k}(n)$. The Möbius function $\mu(n)$ is squarefree and thus lemma 2.2 may be used. Moreover, $\mu(p)=-1$, $\prod_{p^\alpha \| d}(-1)^\alpha f^\alpha(p)=1$ and therefore

$$\Lambda_{\mu,k}(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)\log^k\frac{n}{d}=\sum_{d|n}\mu(d)\log^k d. \tag{13}$$

Since $\Lambda_k(n)=\sum_{d|n}\mu(d)\log^k\frac{n}{d}$, which is similar to (13), it might be expected that $\Lambda_{\mu,k}(n)$ could be expressed in some way by the functions $\Lambda_k(n)$. This is indeed so as

$$\Lambda_{\mu,k}(n)=\sum_{d|n}\mu(d)\log^k d=\sum_{d|n}\mu(d)\left(\log n-\log\frac{n}{d}\right)^k=$$

$$\sum_{d|n}\mu(d)\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\log^i n\cdot\log^{k-i}\frac{n}{d}=$$

$$\sum_{d|n}\mu(d)\sum_{i=0}^{k-1}(-1)^{k-i}\binom{k}{i}\log^i n\cdot\log^{k-i}\frac{n}{d}+\sum_{d|n}\mu(d)\log^k n=$$

$$\sum_{i=0}^{k-1}(-1)^{k-i}\binom{k}{i}\log^i n\sum_{d|n}\mu(d)\log^{k-i}\frac{n}{d}=$$

$$\sum_{i=0}^{k-1}(-1)^{k-i}\binom{k}{i}\Lambda_{k-i}(n)\log^i n.$$

This gives for $k=1$

$$\Lambda_{\mu,1}(n)=\Lambda_\mu(n)=-\Lambda_1(n)=-\Lambda(n), \tag{14}$$

where $\Lambda(n)$ is the von Mangoldt function.

Evaluation of $\Lambda_{\chi,k}(n)$. Since all characters $\chi(n)$ for a given modulus i are totally multiplicative functions, (11) gives immediately

$$\Lambda_{\chi,k}(n)=\chi(n)\Lambda_k(n). \tag{15}$$

Evaluation of $\Lambda_{\tau,k}(n)$. $\tau(n)=\sum_{d|n} 1$ is the number of divisors function. If we use theorem 1.4 with $g(n)=h(n)=1$ then

$$\Lambda_{\tau,k}(n)=2\Lambda_k(n)+\sum_{i=1}^{k-1}\binom{k}{i}\sum_{d|n}\Lambda_i(d)\,\Lambda_{k-i}\left(\frac{n}{d}\right). \tag{16}$$

Evaluation of $\Lambda_{\sigma_i,k}(n)$. $\sigma_i(n)=\sum_{d|n} d^i=\sum_{d|n}\left(\frac{n}{d}\right)^i$ is the sum of divisor powers function, where $i>0$. Theorem 1.4 can be used with $g(n)=1$, $h(n)=n^i$, $\Lambda_{h,k}(n)=n^i\Lambda_k(n)$ (since $h(n)$ is totally multiplicative) to give

$$\Lambda_{\sigma_i,k}(n)=\Lambda_k(n)\,(1+n^i)+\sum_{j=1}^{k-1}\binom{k}{j}\sum_{d|n}\Lambda_{k-j}(d)\,\Lambda_j\left(\frac{n}{d}\right)\left(\frac{n}{d}\right)^i. \tag{17}$$

Evaluation of $\Lambda_{\phi,k}(n)$. Euler's totient function $\phi(n)$ can be expressed as $\phi(n)=\sum_{d|n}\mu(d)\frac{n}{d}$, and therefore theorem 1.4 may be used with $g(n)=\mu(n)$, $h(n)=n$ to obtain

$$\Lambda_{\phi,k}(n)=\Lambda_{\mu,k}(n)+n\Lambda_k(n)+\sum_{i=1}^{k-1}\binom{k}{i}\sum_{d|n}\Lambda_{\mu,i}(d)\,\Lambda_{k-i}\left(\frac{n}{d}\right)\frac{n}{d}. \tag{18}$$

Evaluation of $\Lambda_{r,k}(n)$. The function $r(n)$ is the number of representations of n as a sum of two squares. It is known ([2], pp. 241—242) that $r(n)=4\sum_{d|n}\chi(d)$ where

$$\chi(n)=\begin{cases}(-1)^{\frac{n(n-1)}{2}} & n=2k+1\\ 0 & n=2k\end{cases}$$

and the constant 4 comes from all possible combinatitions of signs when $n=x^2+y^2$. The function $\chi(n)$ is the non-principal character mod 4 and therefore $\Lambda_{\chi,k}(n)=\chi(n)\Lambda_k(n)$. If we put $r(n)=4s(n)$ where $s(n)=\sum_{d|n}\chi(d)$ then

$$\sum_{n=1}^{\infty}\Lambda_{r,k}(n)\,n^{-s}=(-1)^k\frac{R^{(k)}(s)}{R(s)}=(-1)^k\frac{\left(\sum_{n=1}^{\infty}r(n)\,n^{-s}\right)^{(k)}}{\sum_{n=1}^{\infty}r(n)\,n^{-s}}=$$

$$(-1)^k\frac{\left(\sum_{n=1}^{\infty}4s(n)\,n^{-s}\right)^{(k)}}{\sum_{n=1}^{\infty}4s(n)\,n^{-s}}=(-1)^k\frac{S^{(k)}(s)}{S(s)}=\sum_{n=1}^{\infty}\Lambda_{s,k}(n)\,n^{-s},$$

so that $\Lambda_{r,k}(n)=\Lambda_{s,k}(n)$. For $s(n)$ theorem 1.4 may be used with $g(n)=1$, $h(n)=\chi(n)$ and therefore

$$(19)\quad \Lambda_{r,k}(n)=\Lambda_{s,k}(n)=(1+\chi(n))\Lambda_k(n)+\sum_{i=1}^{\infty}\binom{k}{i}\sum_{d|n}\Lambda_i(d)\Lambda_{k-i}\left(\frac{n}{d}\right)\chi\left(\frac{n}{d}\right).$$

3. Asymptotic formulas.

This section contains asymptotic formulas for $\sum_{n\le x}\Lambda_{f,k}(n)$ when $f(n)$ is $\tau(n)$, $\mu(n)$ or $\phi(n)$, and a general theorem about $\sum_{n\le x}\Lambda_{f,k}(n)$ with applications to functions connected with the two-square problem and to the function $\tau_m(n)$ which is the number of representations of n as a product of m factors.

Theorem 3.1 $\sum_{n\le x}\Lambda_{\tau,k}(n)=k(k+1)x\log^{k-1}x+O(x\log^{k-1}x)$.

Proof. Using (16) and formulas

$$(20)\qquad \sum_{n\le x}\Lambda_k(n)=kx\log^{k-1}x+O(x\log^{k-2}x),$$

$$(21)\qquad \sum_{mn\le x}\Lambda_r(m)\Lambda_s(n)=\frac{r!\,s!}{(r+s-1)!}x\log^{r+s-1}x+O(x\log^{r+s-2}x)$$

proved in [3] we have

$$\sum_{n\le x}\Lambda_{\tau,k}(n)=\sum_{n\le x}2\Lambda_k(n)+\sum_{i=1}^{k-1}\binom{k}{i}\sum_{n\le x}\sum_{d|n}\Lambda_i(d)\Lambda_{k-i}\left(\frac{n}{d}\right)=$$

$$2kx\log^{k-1}x+O(x\log^{k-2}x)+\sum_{i=1}^{k-1}\binom{k}{i}\sum_{mn\le x}\Lambda_i(m)\Lambda_{k-i}(n)=$$

$$\sum_{i=1}^{k-1}\frac{k!\,i!\,(k-i)!\,x\log^{k-1}x}{(k-i)!\,i!\,(k-1)!}+2kx\log^{k-1}x+O(x\log^{k-2}x)=$$

$$2kx\log^{k-1}x+\sum_{i=1}^{k-1}kx\log^{k-1}x+O(x\log^{k-2}x)=$$

$$k(k+1)x\log^{k-1}x+O(x\log^{k-2}x).$$

Theorem 3.2. $\sum_{n\le x}\Lambda_{\mu,k}(n)=\begin{cases}-x+O\left(\dfrac{x}{\log x}\right) & k=1\\ O(x\log^{k-2}x) & k\ge 2\end{cases}$.

Proof. Partial summation of (20) gives

$$(22)\qquad \sum_{n\le x}\Lambda_k(n)\log^i n=kx\log^{k+i-1}x+O(x\log^{k+i-2}x).$$

Using (22) and the formula $\Lambda_{\mu,k}(n)=\sum_{i=0}^{k-1}(-1)^{k-i}\binom{k}{i}\Lambda_{k-i}(n)\log^i n$ with $k-i$ instead of k we shall have

$$\sum_{n\leq x}\Lambda_{\mu,k}(n)=\sum_{n\leq x}\sum_{i=0}^{k-1}(-1)^{k-1}\Lambda_{k-i}(n)\log^i n=$$

$$\sum_{i=0}^{k-1}(-1)^{k-i}\binom{k}{i}[(k-i)\,x\log^{k-1}x+O(x\log^{k-2}x)]=$$

$$\left[\sum_{i=0}^{k-1}(-1)^{k-i}(k-i)\binom{k}{i}\right]x\log^{k-1}x+O(x\log^{k-2}x)=$$

$$\sum_{j=1}^{k}(-1)^j\binom{k}{j}jx\log^{k-1}x+O(x\log^{k-2}x)=$$

$$\left.\frac{d(1-t)^k}{dt}\right|_{t=1}\cdot x\log^{k-1}x+O(x\log^{k-2}x)=\begin{cases}-x+O\left(\dfrac{x}{\log x}\right) & k=1\\ O(x\log^{k-2}x) & k\geq 2\end{cases}$$

because

$$\left.\frac{d(1-t)^k}{dt}\right|_{t=1}=\begin{cases}-1 & k=1\\ 0 & k\geq 2\end{cases}.$$

Theorem 3.3. $\sum_{n\leq x}\Lambda_{\phi,k}(n)=\dfrac{kx^2}{2}\log^{k-1}x+O(x^2\log^{k-2}x)$.

Proof. Using (18) we have

$$\sum_{n\leq x}\Lambda_{\phi,k}(n)=\sum_{n\leq x}\Lambda_{\mu,k}(n)+\sum_{n\leq x}n\Lambda_k(n)+\sum_{i=1}^{k-1}\binom{k}{i}\sum_{n\leq x}\sum_{d|n}\Lambda_{\mu,i}(d)\,\Lambda_{k-i}\left(\frac{n}{d}\right)\frac{n}{d}.$$

For $k\geq 2$ $\sum_{n\leq x}\Lambda_{\mu,k}(n)=O(x\log^{k-2}x)$ and thus for $i=1,\ldots,k-1$

$$\sum_{n\leq x}\sum_{d|n}\Lambda_{\mu,i}(d)\,\Lambda_{k-i}\left(\frac{n}{n}\right)\frac{n}{d}=\sum_{mn\leq x}\Lambda_{\mu,i}(m)\,\Lambda_{k-i}(n)\,n=$$

$$\sum_{n\leq x}n\Lambda_{k-i}(n)\sum_{m\leq x/n}\Lambda_{\mu,i}(m)=\sum_{n\leq x}n\Lambda_{k-i}(n)\,O\left(\frac{x}{n}\log^{i-1}\frac{x}{n}\right)=$$

$$O\left(x\log^{i-1}x\sum_{n\leq x}\Lambda_{k-i}(n)\right)=O(x\log^{i-1}x\cdot x\log^{k-i-1}x)=$$

$$O(x^2\log^{k-2}x).$$

Partial summation of (20) gives

$$\sum_{n\leq x} n\Lambda_k(n) = \frac{kx^2}{2}\log^{k-1} x + O(x^2\log^{k-2} x),$$

which proves the theorem for $k\geq 2$, but since $\Lambda_{\Phi,1}(n) = (n-1)\Lambda_1(n)$ it is easily seen that the theorem is also true when $k=1$.

Before passing on to theorem 3.4, it is necessary to prove the following lemma.

Lemma 3.1 *Let $f(n)$ be a non-negative (not necessarily multiplicative) arithmetical function and let for $m\geq 2$*

$$\sum_{n\leq x} f(n) = A\log^m x + O(\log^{m-1} x),$$

where $A\neq 0$. Then

$$\sum_{n\leq x} \frac{f(n)}{\log\frac{2x}{n}} = O(\log^{m-1} x).$$

Proof. Since by hypothesis we have

$$\sum_{n\leq x} \frac{f(n)}{\log x} = A\log^{m-1} x + O(\log^{m-2} x) = O(\log^{m-1} x),$$

it is sufficient to show that

$$\sum_{n\leq x} f(n)\left(\frac{1}{\log\frac{2x}{n}} - \frac{1}{\log x}\right) = O(\log^{m-1} x).$$

$$\sum_{n\leq x} f(n)\left(\frac{1}{\log\frac{2x}{n}} - \frac{1}{\log x}\right) = \sum_{n\leq x} f(n)\int_{2x/n}^{x}\frac{dt}{t\log^2 t} =$$

$$O(1) + \int_2^x \frac{\sum\limits_{2x/t\leq n\leq x} f(n)}{t\log^2 t}\,dt =$$

$$O(1) + \int_2^x \frac{A\log^m x + O(\log^{m-1} x) - A\log^m 2x/t + O(\log^{m-1} 2x/t)}{t\log^2 t}\,dt =$$

$$O(1) + \int_2^x \frac{O(\log^{m-1} x) + C\log x\cdot\log^{m-1} t + O(\log^{m-1} x)}{t\log^2 t}\,dt = O(\log^{m-1} x).$$

Theorem 3.4 *Suppose* $\Lambda_{f,1}(n)\geq 0$ *and*

$$\sum_{n\leq x}\Lambda_{f,1}(n)=Ax+O\left(\frac{x}{\log x}\right), \tag{23}$$

where $A\neq 0$. *Then*

$$\sum_{n\leq x}\Lambda_{f,k}(n)=A\prod_{i=1}^{k-1}\left(1+\frac{A}{i}\right)\cdot x\log^{k-1}x+O(x\log^{k-2}x). \tag{24}$$

Proof. Induction on k. Since (24) reduces to (23) when $k=1$, the theorem is true for $k=1$. (23) in fact generalizes the prime number theorem, since for $A=1$, $f(n)=1$ we obtain

$$\sum_{n\leq x}\Lambda_{1,1}(n)=\sum_{n\leq x}\Lambda(n)=x+O\left(\frac{x}{\log x}\right),$$

which is a version of the prime number theorem (see chs. I and III of [1]) with a weak error term. Thus the hypothesis made in (23) is a natural one, since many arithmetical functions have asymptotic distributions similar to the distribution of primes.

Suppose now the theorem is true for some k, then by (8)

$$\Lambda_{f,k+1}(n)=\Lambda_{f,k}(n)\log n+\sum_{d|n}\Lambda_{f,k}(d)\Lambda_{f,1}\left(\frac{n}{d}\right),$$

which shows that $\Lambda_{f,1}(n)\geq 0$ implies $\Lambda_{f,k}(n)\geq 0$ for all k. Summation on n gives

$$\sum_{n\leq x}\Lambda_{f,k+1}(n)=\sum_{n\leq x}\Lambda_{f,k}(n)\log n+\sum_{n\leq x}\sum_{d|n}\Lambda_{f,k}(d)\Lambda_{f,1}\left(\frac{n}{d}\right)=$$
$$\sum_{n\leq x}\Lambda_{f,k}(n)\log n+\sum_{n\leq x}\Lambda_{f,k}(n)\sum_{m\leq x/n}\Lambda_{f,1}(m).$$

Since for $x\geq 2$ $\dfrac{1}{\log x}\leq\dfrac{2}{\log 2x}$, (23) may be written as

$$\sum_{n\leq x}\Lambda_{f,1}(n)=Ax+O\left(\frac{x}{\log 2x}\right), \tag{25}$$

which is easier to work with because $\dfrac{1}{\log\frac{2x}{n}}$ stays bounded for $n=x$, while $\dfrac{1}{\log\frac{x}{n}}$ does not.

Partial summation and induction hypothesis give

$$\sum_{n\leq x}\Lambda_{f,k}(n)\log n=A\prod_{i=1}^{k-1}\left(1+\frac{A}{i}\right)\cdot x\log^{k}x+O(x\log^{k-1}x), \tag{26}$$

$$\sum_{n\leq x}\frac{\Lambda_{f,k}(n)}{n}=\frac{A}{k}\prod_{i=1}^{k-1}\left(1+\frac{A}{i}\right)\log^{k}x+O(\log^{k-1}x). \tag{27}$$

Therefore

$$\sum_{n\le x}\Lambda_{f,k}(n)\sum_{m\le x/n}\Lambda_{f,1}(m)=\sum_{n\le x}\Lambda_{f,k}(n)\left[\frac{Ax}{n}+O\left(\frac{x}{n\log\frac{2x}{n}}\right)\right]=$$

$$\frac{A^2}{k}\cdot\prod_{i=1}^{k-1}\left(1+\frac{A}{i}\right)\cdot x\log^k x+O(x\log^{k-1}x)+O\left(x\sum_{n\le x}\frac{\Lambda_{f,k}(n)}{n\log\frac{2x}{n}}\right)=$$

$$\frac{A^2}{k}\cdot\prod_{i=1}^{k-1}\left(1+\frac{A}{i}\right)\cdot x\log^k x+O(x\log^{k-1}x), \tag{28}$$

where lemma 3.1 was used with $f(n)=\frac{1}{n}\Lambda_{f,k}(n)$, $m=k$ Addition of (26) and (28) gives

$$\sum_{n\le x}\Lambda_{f,k+1}(n)=$$

$$A\prod_{i=1}^{k-1}\left(1+\frac{A}{i}\right)\cdot x\log^k x+\frac{A^2}{k}\cdot\prod_{i=1}^{k-1}\left(1+\frac{A}{i}\right)\cdot x\log^k x+O(x\log^{k-1}x)=$$

$$A\prod_{i=1}^{k}\left(1+\frac{A}{i}\right)\cdot x\log^k x+O(x\log^{k-1}x),$$

which ends the proof of theorem 3.4.

Consider first the application of theorem 3.4 to the function $b(n)$, the characteristic function of numbers that are a sum of two squares:

$$b(n)=\begin{cases}1 & n=x^2+y^2\\ 0 & n\ne x^2+y^2\end{cases}.$$

The function $b(n)$ is multiplicative, and from the formula $r(n)=4\sum_{d\mid n}\chi(d)$ it is easily seen that the multiplicative semigroup of numbers that are a sum of two squares if generated by 2, by primes of the form $4k+1$ and by squares of the primes of the form $4k+3$. Therefore

$$B(s)=\sum_{n=1}^{\infty}b(n)n^{-s}=(1-2^{-s})^{-1}\prod_{p\equiv 1\,(\mathrm{mod}\,4)}(1-p^{-s})^{-1}\prod_{p\equiv 3\,(\mathrm{mod}\,4)}(1-p^{-2s})^{-1}.$$

$$\log B(s)=-\log(1-2^{-s})-\log\prod_{p\equiv 1\,(\mathrm{mod}\,4)}(1-p^{-s})-\log\prod_{p\equiv 3\,(\mathrm{mod}\,4)}(1-p^{-2s}),$$

$$\sum_{n=1}^{\infty}\Lambda_{b,1}(n)n^{-s}=-\frac{B'(s)}{B(s)}=-[\log B(s)]'=$$

$$\sum_{i=0}^{\infty}\frac{\log 2}{2^{is}}+\sum_{p\equiv 1\,(\mathrm{mod}\,4)}\sum_{i=0}\frac{\log p}{p^{is}}+\sum_{p\equiv 3\,(\mathrm{mod}\,4)}\sum_{i=0}\frac{2\log p}{p^{2is}},$$

so that

$$\Lambda_{b,1}(n)=\begin{cases}\log p & n=p^i \quad p=2, \quad p\equiv 1 \pmod 4\\ 2\log p & n=p^{2i} \qquad\qquad p\equiv 3 \pmod 4\end{cases}.$$

$$\sum_{n\le x}\Lambda_{b,1}(n)=\sum_{2^i\le x}\log 2+\sum_{\substack{p^i\le x\\ p\equiv 1 \pmod 4}}\log p+\sum_{\substack{p^{2i}\le x\\ p\equiv 3 \pmod 4}}\log p=$$

$$\frac{x}{2}+O\left(\frac{x}{\log x}\right),$$

where the theorem for prime numbers in arithmetical progressions ([7], p. 157) and partial summation are used.

Theorem 3.4 may now be applied with $A=\frac{1}{2}$ so that

$$\sum_{n\le x}\Lambda_{b,k}(n)=\frac{1}{2}\prod_{i=1}^{k-1}\left(1+\frac{1}{2i}\right)x\log^{k-1}x+O(x\log^{k-2}x)=\tag{29}$$

$$\frac{(2k-1)!!}{2^k(k-1)!}x\log^{k-1}x+O(x\log^{k-2}x).$$

To obtain the asymptotic formula for $\sum_{n\le x}\Lambda_{r,k}(n)$ note that by (19)

$$\Lambda_{r,1}(n)=\Lambda(n)+\chi(n)\Lambda(n)$$

so that $\Lambda_{r,1}(n)\ge 0$ since

$$\chi(n)\Lambda(n)=\begin{cases}0 & n\ne p^i\\ 0 & n=2^i\\ \log p & n=p^i \quad p\equiv 1 \pmod 4\\ (-1)^i\log p & n=p^i \quad p\equiv 3 \pmod 4\end{cases}.$$

$$\sum_{n\le x}\Lambda_{r,1}(n)=\sum_{n\le x}\Lambda(n)+\sum_{n\le x}\chi(n)\Lambda(n)=$$

$$x+O\left(\frac{x}{\log x}\right)+\sum_{\substack{p\le x\\ p\equiv 1 \pmod 4}}\log p+\sum_{\substack{p\le x\\ p\equiv 3 \pmod 4}}(-\log p)=$$

$$x+O\left(\frac{x}{\log x}\right)+\left(\frac{x}{2}+O\left(\frac{x}{\log x}\right)\right)+\left(-\frac{x}{2}+O\left(\frac{x}{\log x}\right)\right)=$$

$$x+O\left(\frac{x}{\log x}\right),$$

where the prime number theorem for primes in an arithmetical progression was used again. Theorem 3.4 gives then ($A=1$)

$$\sum_{n\le x}\Lambda_{r,k}(n)=kx\log^{k-1}x+O(x\log^{k-2}x).\tag{30}$$

Finally, let $\tau_m(n) = \sum_{a_1 a_2 \cdots a_m = n} 1$ be the number of representations of n as a product of m factors (of which any may be unity). Since ([2], p. 255) $\zeta^m(s) = \sum_{n=1}^{\infty} \tau_m(n)\, n^{-s}$ we have

$$\sum_{n=1}^{\infty} \Lambda_{\tau_m, 1}(n)\, n^{-s} = [-\log \zeta^m(s)]' = -m \frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} m \Lambda(n)\, n^{-s}$$

so that

$$\Lambda_{\tau_m, 1}(n) = m \Lambda(n).$$

The hypotheses of theorem 3.4 are satisfied since $\Lambda(n) \geq 0$,

$$\sum_{n \leq x} \Lambda_{\tau_m, 1}(n) = m \sum_{n \leq x} \Lambda(n) = mx + O\left(\frac{x}{\log x}\right),$$

so that we obtain with $A = m$

Theorem 3.5. $\sum_{n \leq x} \Lambda_{\tau_m, k}(n) = m \prod_{i=1}^{k-1} \frac{m+i}{i} \cdot x \log^{k-1} x + O(x \log^{k-2} x).$

If we set $m = 2$ we get another proof of theorem 3.1 since

$$2 \prod_{i=1}^{k-1} \frac{2+i}{i} = k(k+1), \quad \text{and} \quad \tau_2(n) = \sum_{a_1 a_2 = n} 1 = \sum_{d \mid n} 1 = \tau(n).$$

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