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COINCIDENCE DEGREE AND RABINOWITZ'S BIFURCATION THEOREM

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1. Introduction

In [17] Rabinowitz investigates the structure of solutions of a nonlinear eigenvalue problem of the form $x=G(\lambda, x)$ where $G:\mathbf{R}\times X\rightarrow X$ is a completely continuous, in 0 Fréchet-differentiable operator and X a Banach space. He demonstrates the existence of continua of solutions and gives an alternative description of them. Applications are given to nonlinear Sturm-Liouville problems for second-order ordinary differential equations, to nonlinear eigenvalue problems for a class of quasilinear elliptic partial differential equations on bounded domains in $\mathbf{R}^n$, and to Hammerstein integral equations with oscillation kernels ([17], [18], [19]).

In dealing with differential equations on unbounded domains the assumption of a completely continuous nonlinearity has to be relaxed. Thus Stuart [10] considers the operator equation $x=\lambda Nx$, where $N:X\rightarrow X$ is a k-set-contraction.

On the other hand the study of nonlinear Steklov problems leads Stuart and Toland [22] to an extension of Rabinowitz's result to the case, where the derivate of G has the form $A_1+\lambda A_2$.

In his thesis Laloux [10] shows, that some technical assumptions which Stuart and Toland had to suppose, can be avoided by using an extension of Rabinowitz's result for a bifurcation equation $Lx=N(\lambda, x)$, where X, Y are Banach spaces, $L:X\supseteq D(L)\rightarrow Y$ is a Fredholm operator with index 0 and $N:X\rightarrow Y$ is completely continuous. Some other applications are also given.

In proving their alternative results Rabinowitz and Stuart use the Leray-Schauder respectively the Nussbaum degree. The extension, given by Laloux, is based on a coincidence degree for completely continuous nonlinearities, established by J. Mawhin in [14]. Using coincidence degree for k-set-contractions [5], we derive Stuart's result for operator equations $Lx=\lambda Nx$, where L is like above, but N is a k-set-contraction (Section 3. and 4.).

In Section 5. we reduce as an application a boundary value problem for a nonlinear functional differential equation of neutral type to an eigenvalue problem for a linear functional integral equation. In Section 2. we collect some later needed definitions and results.

2. Preliminaries

Let X be a real Banach space. The set-measure of noncompactness γ of a subset M of X is defined by:

$\gamma(M) := \inf\{\varepsilon \mid \varepsilon > 0$, there exists a finite covering of M by subsets of X with diameter lower than $\varepsilon\}$.

Assume that Y is a further real Banach space, $\Omega \subseteq X$ and $k \in \mathbf{R}^+$, then a continuous function $f: \Omega \to Y$ is called k-set-contraction, if $\gamma(f(M)) \leq k\gamma(M)$ for each bounded subset M of Ω. Usually a 0-set-contraction is said to be completely continuous (compact). We introduce the following notation: $D(f)$ means the domain of f, $R(f)$ the range of f. If $L: X \supseteq D(L) \to Y$ is a linear operator, $\operatorname{Ker}(L)$ denotes the null space of L, and we set:

$$\alpha(L) := \dim(\operatorname{Ker}(L)), \quad \beta(L) := \dim(Y/R(L)),$$

where $\alpha(L) = \infty$ respectively $\beta(L) = \infty$ in the infinite dimensional case:

L is called a Φ_+-operator, if L is closed, $R(L)$ is closed, and $\alpha(L) < \infty$, and a Fredholm operator, if additionally $\beta(L) < \infty$. If L is a Fredholm operator, $\operatorname{ind}(L) := \alpha(L) - \beta(L)$ means the (Fredholm-) index of L. Further we set:

$$1(L) := \sup\{r \mid r \in \mathbf{R}^+,\ r\gamma(M) \leq \gamma(L(M)) \text{ for each bounded } M \subseteq D(L)\}$$

In [5] is shown that for a closed operator L, $1(L) > 0$, if and only if L is a Φ_+-operator. Basic for the sequel is the concept of coincidence degree, given in [5]. We sketch the here needed case; we assume:

(a) X, Y are Banach spaces, $L: X \supseteq D(L) \to Y$ is a Fredholm operator, and $\operatorname{ind}(L) = 0$.

(b) $\Omega \subseteq X$ is open and bounded, $\Omega \cap D(L) \neq \Phi$, $N: \overline{\Omega} \to Y$ is a k-set--contraction with $0 \leq k < 1(L)$.

(c) $Lx \neq Nx$ for $x \in \partial\Omega \cap D(L)$, where $\partial\Omega$ denotes the boundary of Ω.

From (a) we deduce the existence of continuous projectors $P: X \to X$, $Q: Y \to Y$ with $R(P) = \operatorname{Ker}(L)$ and $\operatorname{Ker}(Q) = R(L)$, and of a linear isomorphism $J: R(Q) \to R(P)$. Further we set $L_P := L \mid \operatorname{Ker}(P) \cap D(L)$ and remark that L_P^{-1} is continuous according to (a). Finally let $M_J: \overline{\Omega} \to X$ be defined by:

$$M_J := P + J \circ Q \circ N + L_P^{-1} \circ (I - Q) \circ N$$

then M_J is a $k/1(L)$-set-contraction and $0 \notin (I - M_J)(\partial\Omega)$, since (b) and (c) are satisfied. Therefore we can define the coincidence degree of (L, N) by:

$$D_J[(L, N), \Omega] := \deg(I - M_J, \Omega, 0)$$

where deg denotes the degree for k-set-contractions with $0 \leq k < 1$, [15]. $D_J[(L, N), \Omega]$ is independent of P and Q and has the following properties;

(1) $D_J[(L, N), \Omega] \neq 0$ implies that there exists an

$x \in \Omega \cap D(L)$ with $Lx = Nx$.

(2) If Ω_1, Ω_2 are open subsets of Ω with $\Omega_1 \cap \Omega_2 = \varnothing$

and $\{x \mid x \in \Omega \cap D(L),\ Lx = Nx\} \subseteq \Omega_1 \cup \Omega_2$, then:

$D_J[(L, N), \Omega] = D_J[(L, N), \Omega_1] + D_J[(L, N), \Omega_2]$.

(3) If $\tilde{N}: [0, 1] \times \overline{\Omega} \to Y$ is a k-set-contraction with

$Lx \neq \tilde{N}(\lambda, x)$ for each $x \in \partial\Omega \cap D(L)$ and $\lambda \in [0, 1]$, then

$D_J[(L, \tilde{N}(0, \circ)), \Omega] = D_J[(L, N(1, \circ)), \Omega]$.

For convenience we set: $D[(L, N), \Phi] = 0$. Then one obtains the following, later needed homotopy property from proposition 7 in [15]:

Lemma 1: *Assume that* (a) *is satisfied, I is a compact interval in* $\mathbf{R}$ *and $\Phi \neq \Omega \subseteq I \times X$ is open and bounded in $\mathbf{R} \times X$ with $pr_2(\Omega) \cap D(L) \neq \Phi$, where $pr_2: \mathbf{R} \times X \to X$ denotes the projection on X. Let $F: \overline{\Omega} \to Y$ be continuous and $k \in [0, 1(L))$ with: $\gamma(F(B)) \leq k\gamma(pr_2(B))$ for each $B \subseteq \overline{\Omega}$. Furthermore set $\Omega_t := \{x \mid (t, x) \in \Omega\}$ for $t \in I$. Then $D[(L, F(t, \circ)), \Omega_t]$ is independent of $t \in I$, provided that $Lx \neq F(t, x)$ for all $(t, x) \in \partial\Omega \cap D(L)$.*

We end this section with the definition of a coincidence index. Assume that (a) and (b) hold, but Ω is not necessary bounded, and that $\bar{x} \in \Omega \cap D(L)$ is an isolated coincidence point of (L, N). For each $B_\rho := \{x \mid x \in X, \| x - \bar{x} \| \leq \rho\}$ with $Lx \neq Nx$ for each $\bar{x} \neq x \in B_\rho \cap D(L)$ $D[(L, N), \overset{\circ}{B}_\rho]$ is defined and independent of ρ. Therefore we can define the coincidence index by:

$$i[(L, N), \bar{x}] := D[(L, N), \overset{\circ}{B}_\rho]$$

Additivity and homotopy property can be deduced, using the corresponding assertions for the coincidence degree.

3. Rabinowitz's Theorem for (L, N)

We recall the following definitions:

Definition 1: Let X, Y be Banach spaces, and $L: X \supseteq D(L) \to Y$ and $A: X \to Y$ be linear, then $\lambda \in \mathbf{R}$ is called a characteristic value of (L, A), if $\operatorname{Ker}(L - \lambda A) \neq \{0\}$, otherwise regular.

Definition 2: Let X, Y be Banach spaces, $L: X \supseteq D(L) \to Y$ be linear, and $N: \mathbf{R} \times X \to Y$ continuous with $N(\lambda, 0) = 0$ for each $\lambda \in \mathbf{R}$, then $(\mu, 0)$ is called a bifurcation point, if for every $\varepsilon > 0$ there exists $(\lambda, x) \in \mathbf{R} \times X$ with: $|\lambda - \mu| < \varepsilon$, $0 < \|x\| < \varepsilon$, and $Lx = N(\lambda, x)$.

In the sequel we suppose:

$(H\,1)$ X, Y are Banach spaces over $\mathbf{R}$, $L: X \supseteq D(L) \to Y$

is a Fredholm operator with $\operatorname{ind}(L) = 0$.

$(H2)$ $N: X \to Y$ is a k-set-contraction with $k \in [0, 1(L))$,

$A, B: X \to Y$ are continuous maps with:

$$A \text{ linear, } \lim_{\|x\| \to 0} \frac{\|Bx\|}{\|x\|} = 0, \text{ and } N = A + B.$$

Under these assumptions we consider the bifurcation problem $Lx = \lambda Nx$. The restriction, concerning λ in regard to [17], [18] and [11], is caused by the fact, that the contraction constant is dependent on λ in contrast to the case, where N is compact.

We need the following assertions:

Lemma 2: *Let $(H1)$ be satisfied, $A: X \to Y$ be a linear k-set-contraction with $k \in [0, 1(L))$, and $\mu \in \left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$[1] be regular. Then the set of characteristic values in each compact subinterval of $\left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$ is finite.*

Proof: The case $k = 0$ (i.e. A is completely continuous) is proven in [13]. Hence assume $k \in (0, 1(L))$. Since μ is regular, $L - \mu A$ is injective. Therefore $\lambda \in \mathbf{R}$ is a characteristic value of (L, A), if and only if λ is a characteristic value of

$$(L - \mu A)^{-1} \circ (L - \lambda A) = I - (\lambda - \mu)(L - \mu A)^{-1} \circ A.$$

Now assume that $\varepsilon > 0$ is sufficiently small. We show:

Case 1: If $\mu \geqslant 0$, the set of characteristic values in $\left[0, \frac{1(L)}{k} - \varepsilon\right]$ is finite, and there is a regular value of (L, A) in $\left[-\frac{1(L)}{k} + \varepsilon, 0\right]$. Since $1(L - \mu A) \geqslant 1(L) - \mu k$ ([5] Theorem 2), $T := \left(\frac{1(L)}{k} - \varepsilon - \mu\right)(L - \mu A)^{-1} \circ A$ is an α-set-contraction with $\alpha = \left(\frac{1(L)}{k} - \varepsilon - \mu\right) k \,(1(L) - \mu k)^{-1} < 1$. Using [1] Theorem 12, we obtain: The set of characteristic values in $[-1, 1]$ of (I, T) is finite, hence the set of (L, A) in $\left[-\frac{1(L)}{k} + \varepsilon + 2\mu, \frac{1(L)}{k} - \varepsilon\right]$. If $-2\mu > -\frac{1(L)}{k} + \varepsilon$, case 1 is proved. Otherwise we choose a regular value μ_1 with; $\mu_1 \leqslant \mu - \frac{1}{2}\left(\frac{1(L)}{k} - \varepsilon - \mu\right) = -\frac{1}{2}\left(\frac{1(L)}{k} - \varepsilon - 3\mu\right)$, and repeat the above described process. Then we receive: (L, A) has only a finite number of characteristic values in $\left[-\frac{1(L)}{k} + \varepsilon + 2\mu_1, \frac{1(L)}{k} - \varepsilon\right]$. Since $-\frac{1(L)}{k} + \varepsilon + 2\mu - \left(-\frac{1(L)}{k} + \varepsilon + 2\mu_1\right) \geqslant \frac{1(L)}{k} - \varepsilon - \mu > 0$ we obtain case 1 after a finite number of steps.

[1] For $k = 0$ we put $1(L)/k = \infty$

Case 2: If $\mu \leqslant 0$, the set of characteristic values in $\left[-\frac{1(L)}{k}+\varepsilon, 0\right]$ is finite and there exist a regular value of (L, A) in $\left(0, \frac{1(L)}{k}-\varepsilon\right]$.

Obviously this assertion can be reduced to case 1 by considering $(L, -A)$ and $-\mu$.

Both cases give the conclusion of Lemma 2.

Lemma 3: *Let (H 1) and (H 2) be satisfied, and assume that* $\mu \in \left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$ *is not a characteristic value of* (L, A). *Then* $(\mu, 0)$ *is not a bifurcation point for* (L, N).

Proof: $(H2)$ implies that A is the Fréchet derivate of N in 0. Therefore A is a k-set-contraction according to [16]. Hence $L-\mu A$ is a Fredholm operator with index 0 [7], so $L-\mu A$ is bijective, because μ is not characteristic value of (L, A). Then there exists an $m>0$ with: $\|(L-\mu A)x\| \geqslant m\|x\|$ for $x \in D(L)$. On the other hand there is an open ball K with center 0 and a $c \in \left(0, \frac{m}{2(|\mu|+1)}\right)$ with $\|Bx\| \leqslant c\|x\|$ for $x \in K$. Set $\delta := \frac{1}{\|A\|+c}\left(\frac{m}{2}-|\mu| c\right)$. Then we have for $\lambda \in \mathbf{R}$ with $|\lambda-\mu| \leqslant \delta$ and $x \in K$:

$$\|Lx-\lambda Nx\| = \|Lx-\lambda Ax-\lambda Bx\| = \|Lx-\mu Ax-(\lambda-\mu)Ax-$$
$$-\lambda Bx\| \geqslant m\|x\|-|\lambda-\mu|\,\|A\|\,\|x\|-(|\mu|+\delta)c\|x\| \geqslant$$
$$\geqslant (m-\delta(\|A\|+c)-|\mu| c)\|x\| = \frac{m}{2}\|x\|.$$

So μ is not a bifurcation point.

We set $E := \mathbf{R} \times X$, equipped with the norm $\|(\lambda, x)\| := (|\lambda|^2+\|x\|^2)^{\frac{1}{2}}$ and Γ to be the closure of $\{(\lambda, x) \mid (\lambda, x) \in E.\ x \neq 0.\ Lx=\lambda Nx\}$ in E. We state:

Lemma 4: *Assume that (H 1) and (H 2) are satisfied, and that* μ *is an isolated characteristic value of* (L, A) *in* $\left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$. *If* φ *denotes the component of* $\Gamma \cup \{(\mu. 0)\}$ *in* E *with* $(\mu. 0) \in \varphi$, *suppose that* φ *fulfilles:* φ *is bounded,* φ *contains only* μ *as a characteristic value of* (L, A) *and*

$$\inf\left\{\left|\lambda \pm \frac{1(L)}{k}\right| \,\middle|\, (\lambda, x) \in \varphi\right\} > 0.$$

Then there are a bounded, open subset 0 *of* $\left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right) \times X$, *a* $\rho>0$ *and an* $\eta>0$ *with:*

(1) $\varphi \subseteq 0$.

(2) $\Gamma \cap \partial 0 = \varnothing$

(3) $\overline{pr_1(0)} \subseteq \left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$, *where* $pr_1 : E \rightarrow \mathbf{R}$ *is the projection on* $\mathbf{R}$,

(4) $(\lambda, x) \in 0$ *and* $|\lambda - \mu| \geq \rho$ *imply* $\|x\| \geq \eta$

(5) $\lambda \neq \mu$ *and* $\lambda \in [-\rho + \mu, \rho + \mu]$ *imply* λ *regular.*

The proof of this Lemma follows the argument used in [20] to prove Lemma 1.7. We note that Lemma 2 ensures the finiteness of the set of characteristic values in each compact subset of $\left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$, and that Lemma 3 has to be used instead of proposition 1.5. there. Using the fact, that for a sequence $(\lambda_n, x_n) \in (\mathbf{R} \times X)^N$ with $Lx_n = \lambda_n N x_n$ holds:

$$\gamma(\{x_n \mid n \in \mathbf{N}\}) \leq \frac{1}{1(L)} \gamma(\{Lx_n \mid n \in \mathbf{N}\}) = \frac{1}{1(L)} \gamma(\{\lambda_n N x_n \mid n \in \mathbf{N}\})$$

the compactness of a closed, bounded subset of Γ can be deduced like there.

If $\mu \in \left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$ is an isolated characteristic value of (L, A), one observes that $i[(L, \lambda N), 0]$ is defined for $0 < |\lambda - \mu| < \delta$ and δ sufficient small in regard to Lemma 3. Now we can state the Rabinowitz alternative in the here considered case:

Theorem 1: *Let (H1) and (H2) be satisfied, and*

$$\mu \in \left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$$

be an isolated charateristic value. φ, Γ *are defined as in Lemma 4.*

$$\textit{If} \lim_{\lambda \to \mu -} i[(L, \lambda N), 0] \neq \lim_{\lambda \to \mu +} i[(L, \lambda N), 0],$$

then one of the following assertions holds:

(i) φ *is unbounded.*

(ii) φ *meets* $(\tilde{\mu}, 0)$ *with* $\tilde{\mu} \neq \mu$,

(iii) $\inf\left\{\left|\lambda \pm \frac{1(L)}{k}\right| \,\middle|\, (\lambda, x) \in \varphi\right\} = 0.$

The proof is analogous to the corresponding in [19] and [20]. We indicate it:

If φ does not satisfy (i)–(iii), assertions (1)–(5) of Lemma 4 hold. Define;

$$r(\lambda) := \begin{cases} \frac{1}{2} \inf \{\zeta \mid \zeta > 0, \text{ there is an } x \in D(L) \text{ with:} \\ \qquad Lx = \lambda N x \text{ and } \|x\| = \zeta, \text{ if } \lambda \in [\mu - \rho, \mu + \rho] \\ \frac{1}{2} \eta, \text{ if } \lambda \in \mathbf{R} \setminus [\mu - \rho, \mu + \rho] \end{cases}$$

For each $\varepsilon>0$ there exists an $a>0$ with: $r(\lambda)\geq a$ for $\lambda\in\mathbf{R}\setminus[\mu-\varepsilon, \mu+\varepsilon]$. Let $O_\lambda:=\{x\mid x\in X, (\lambda, x)\in 0\}$ and $K_{r(\lambda)}:=\{x\mid x\in X, \|x\|\leq r(\lambda)\}$, then for $\lambda\neq\mu$ $D[(L, \lambda N), O_\lambda\setminus K_{r(\lambda)}]$ is defined. Choose $c\in\mathbf{R}^+$, such that

$$c<\frac{1(L)}{k} \text{ and } O_{\pm c}=\varnothing.$$

Using Lemma 1 $D[(L, \lambda N), O_\lambda\setminus K_{r(\lambda)}]$ is constant for $\lambda\in[\Lambda, c]$ and $\mu<\Lambda<c$. Hence $D[(L, \lambda N), O_\lambda\setminus K_{r(\lambda)}]=0$ for $\lambda\in(\mu, c)$. The same assertion holds for $\lambda\in-(c, \mu)$. On the other hand $D[(L, \lambda N), O_\lambda]$ is defined for $\lambda\in[\mu-\delta, \mu+\delta]$ and δ sufficient small, and is constant, using Lemma 1. Now property (2) of the coincidence degree involves that $D[(L, \lambda N), \mathring{K}_{r(\lambda)}]$ is constant for $\lambda\neq\mu$ and $\lambda\in[\mu-\delta, \mu+\delta]$. Hence: $\lim_{\lambda\to\mu-} i[(L, \lambda N), 0]=\lim_{\lambda\to\mu+} i[(L, \lambda N), 0]$.

Remarks:

(1) An example for computing the coincidence index of $(L, \lambda N)$ near an isolated characteristic value without using the multiplicity of $(L, \lambda A)$, can be found in [12].

(2) In applications (e. g. Lyapunovs integral power series [9]) it is possible that one has to restrict the nonlinearity N on a subset of X, to get a suitable k-set-contraction. The following assertion holds in this case: Let the assumptions of Theorem 1 be satisfied. but $D(N)=\bar{\Omega}$, where Ω is an open subset of X with $\Omega\cap D(L)\neq\varnothing$, then φ fulfills one of the assertion (i) – (iii) in Theorem 1, or there is an $(\lambda, x)\in\varphi$ with $x\in\partial\Omega$

4. The case of odd multiplicity

We introduce the concept of multiplicity of a characteristic value of (L, A). given by Laloux and Mawhin [13]. The extension, we need is obvious.

In addition to $(H1)$ and $(H2)$ we assume;

$(H3)$ $Ax\notin R(L)$ for $x\in\operatorname{Ker}(L)\setminus\{0\}$.

Then there exists a unique projector $Q_A: Y\to Y$ with $R(Q_A)=A(\operatorname{Ker}(L))$ and $\operatorname{Ker}(Q_A)=R(L)$. For each continuous projector $P: X\to X$ with $R(P)=\operatorname{Ker}(L)$ the set of nonzero characteristic values for (L, A) is equal to the set of characteristic values of (I, K_P), where $K_P:=L_P^{-1}\circ(I-Q_A)\circ A$. Using results of Ambrosetti [1], we obtain, that the characteristic values of (L, A) in $\left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$ are at most countable with possible accumulation points $-\frac{1(L)}{k}$ or $\frac{1(L)}{k}$. For a nonzero characteristic value $\lambda\in\left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$ of (L, A), λK_P is an α-set -contraction with $\alpha<1$ and therefore $\lim_{i\to\infty}\dim(\operatorname{Ker}(I-\lambda K_P)^i)$ is finite.

Hence we can define:

Definition 3: Let $(H1)-(H3)$ be satisfied and K_P be defined like above. If $\lambda \in \left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$ is a characteristic value of (L, A) we call

$$\nu(\lambda) := \begin{cases} \lim_{i\to\infty} \dim(\operatorname{Ker}(I-\lambda K_P)^i) & \text{for } \lambda \neq 0 \\ \dim(\operatorname{Ker}(L)) & \text{for } \lambda = 0 \end{cases}$$

the multiplicity for λ of (L, A).

Remark: In the case where $X=Y$ and $L=I$, $\nu(\lambda)$ is equal to the classical definition of multiplicity of λA.

Using this Definition and following the arguments of the proofs to Theorem 5.1 and Corollary 5.1 in [13] we can state:

Lemma 5: *Let $(H1)-(H3)$ be fulfilled and $\mu_1, \mu_2 \in \left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$ be regular values of (L, A) with $\mu_1 < \mu_2$. Setting $\delta := \sum_{\lambda \in \Lambda} \nu(\lambda)$, where Λ is the set of characteristic values of $(L. A)$ in (μ_1, μ_2), we have*

$$i[(L, \mu_2 A), 0] = (-1)^{\delta}\, i[(L, \mu_1 A), 0].$$

Concerning the proof of this Lemma we have to comment the derivation of the here needed assertion analogous to formula 5.2 in [13]. A correct proof of a product theorem for the Nussbaum degree is unkown to us. But for linear maps one can follow the proof, given by Fenske [4] for the corresponding theorem in his degree theory, using Stuart's and Toland's result [21], concerning the connection between classical multiplicity and fixed point index of a linear α-set-contraction with $\alpha < 1$. Finally the separability assumption in [4] can be dropped in view of the following fact. If X is a Banach space and $T: X \to X$ is a linear α-set-contraction with $0 \leq \alpha < 1$, then $I-T$ is a Fredholm map with index 0, hence $I-T$ surjective, if injective. Therefore each $x \in X$ is regular concerning $I-T$ and one does not need the Smale-Sard-Theorem. We also refer to the remarks in [3].

Using this Lemma we obtain an analogy to Rabinowitz's results for the case of odd multiplicity.

Theorem 2: *Let $(H1)-(H3)$ be satisfied and $\mu \in \left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$ be a characteristic value of (L, A) with odd multiplicity. Then the alternative assertion of Theorem 1 is true.*

Proof: Let $\lambda \in \left(-\frac{1(L)}{k}, \frac{1(L)}{k}\right)$ be a regular value of (L, A). Then Lemma 3 implies, that there exists an $\rho_0 > 0$, such that $Lx \neq \lambda Nx$ for each $x \in D(L)$ with $0 < \|x\| \leqslant \rho_0$. Using $(H2)$, we can further assume, that

$$\|Bx\| \leq \frac{m}{2(|\lambda|+1)} \|x\| \quad \text{for} \quad \|x\| \leq \rho_0,$$

where $m>0$ with: $\|(L-\lambda A)x\| \geqslant m\|x\|$ for $x \in D(L)$ (see proof to Theorem 3). Let $\rho \in (0, \rho_0)$ and set:

$$\tilde{N}(t, x) := t\lambda Nx + (1-t)\lambda Ax$$

for $t \in [0, 1]$ and $x \in K_\rho := \{z \mid z \in X, \|z\| \leq \rho\}$ $\tilde{N}$ is a α-set-contraction with $\alpha = |\lambda| k < 1(L)$. Further $x \in \partial K_\rho \cap D(L)$ involves:

$$\|Lx - \tilde{N}(t, x)\| = \|Lx - t\lambda Nx - (1-t)\lambda Ax\|$$

$$\geqslant m\|x\| - t|\lambda| \frac{m}{2(|\lambda|+1)} \|x\| \geq \frac{m}{2}\|x\|$$

Hence $\tilde{N}$ is an admissible homotopy and therefore

$$D_J[(L, \lambda N), \mathring{K}_\rho] = D_J[(L, \lambda A), \mathring{K}_\rho]$$

for $J: \operatorname{Ker}(L) \to R(Q_A)$ a linear isomorphism and for any $\rho \in (0, \rho_0)$ (property (3) of the coincidence degree). Now Lemma 5 and Theorem 1 imply the assertion of this Theorem.

Remarks:

(1) The special case $X = Y$, $L = I$ is proven by Stuart in [20].

(2) The corresponding result to remark (2) after Theorem 1 is valid.

5. An application to a functional differential equation of neutral type.

In this section we reduce a nonlinear boundary value problem for a functional differential equation of neutral type to an eigenvalue problem of a linear functional integral equation, using Theorem 2.

First we introduce some notations; Let $n \in \mathbf{N}$, $s \in \mathbf{Z}^+$ and $|\cdot|$ a norm on $\mathbf{R}^n$, then $C^s([0, 1], \mathbf{R}^n)$ denotes the Banach space of s-times continuously differentiable $\mathbf{R}^n$-valued functions with domain $[0, 1]$. We set;

$$\|u\|_\infty := \max\{|u(x)| \mid x \in [0, 1]\} \text{ for } u \in C^\circ([0, 1], \mathbf{R}^n),$$

$$\|u\|_s := \max\{\|u^{(j)}\|_\infty \mid 0 \leq j \leq s\} \text{ for } u \in C^s([0, 1], \mathbf{R}^n),$$

and assume $\| \ \|_s$ to be the norm on $C^s([0, 1], \mathbf{R}^n)$.

For $\sigma: [0, 1] \to [0, 1]^n$ and $u \in C^\circ([0, 1], \mathbf{R})$ we set:

$$u \circ \sigma := (u \circ \sigma_1, \ldots, u \circ \sigma_n),$$

where $\sigma_1, \ldots, \sigma_n$ denote the components of σ.

Now we can state the problem we will consider here. Let

$$p \in C^1([0, 1], \mathbf{R}),\ m \in \mathbf{N},\ a = (a_1, \ldots, a_m) \in \mathbf{R}^m \text{ with } \sum_{j=1}^m a_j \neq 0,$$

$\sigma=(\sigma_1, \ldots, \sigma_m) \in C([0, 1], \mathbf{R}^m)$ with $\sigma_1(x)=x$ for $x \in [0, 1]$ and $R(\sigma_j) \subseteq [0, 1]$ for $2 \leq j \leq m$, and $f \in C([0, 1] \times \mathbf{R}^{3m}, \mathbf{R})$. We seek $\lambda \in \mathbf{R}$ and $u \in C^2([0, 1], \mathbf{R})$ with:

$$(BVP) \begin{cases} (p(x)\, u'(x))' = \langle [\lambda a, u \circ \sigma(x) \rangle + f(x, u \circ \sigma(x), u' \circ \sigma(x), u'' \circ \sigma(x))] \\ u(0)=u(1),\ u'(0)=u'(1) \end{cases}$$

where $\langle , \rangle$ denotes the euclidian scalar product of the $\mathbf{R}^m$. We obtain:

Theorem 3: *Suppose that p, m, a. σ and f satisfy the above described assumptions, and that $p_0 := \min\{p(x) \mid x \in [0, 1]\} > 0$ Further assume:*

(1) *There is a $k \in \boldsymbol{R}^+$ with:*

$$|f(x, \zeta_1, \zeta_2, \zeta_3) - f(x, \zeta_1, \zeta_2, \zeta_4)| \leq k\, |\zeta_3 - \zeta_4|$$

for $x \in [0, 1]$, $\zeta_1, \ldots, \zeta_4 \in \mathbf{R}^m$

(2) $f(\cdot, \zeta_1, \zeta_2, \zeta_3)$ *is* $o(|\zeta_1| + |\zeta_2| + |\zeta_3|)$ *for*

$|\zeta_1| + |\zeta_2| + |\zeta_3| \to 0$ *uniformly on* $[0, 1]$

(3) *Let* $h_1(x, z) := \int_z^x \frac{1}{p(y)}\, dy$ *for* $z, x \in [0, 1]$ *and*

$$h_2(x, z) := -x \int_z^1 \frac{1}{p(y)}\, dy - \int_0^x \frac{y}{p(y)}\, dy + x \int_0^1 \frac{y}{p(y)}\, dy$$

for $z, x \in [0, 1]$. We assume, that there is a $\lambda_0 \in \left(-\frac{p_0}{k}, \frac{p_0}{k}\right)$ with: λ_0 is an eigenvalue with odd dimensional generalized eigenspace of the linear Volterra-Hammerstein integral equation:

$$(I) \begin{cases} u(x) = \lambda \int_0^x h_1(x, z) \langle a, u \circ \sigma(z) \rangle dz + \lambda \int_0^1 h_2(x, z) \langle a, u \circ \sigma(z) \rangle dz \\ u \in C^2([0, 1], \mathbf{R}^n);\ u(0)=u(1),\ u'(0)=u'(1) \end{cases}$$

Then there exists a continuum $\varphi \subseteq \mathbf{R} \times C^2([0, 1], \mathbf{R}]$ of solutions of (BVP) with $(\lambda_0, 0) \in \varphi$, which satisfies one of the following assertions:

(i) φ *is unbounded.*

(ii) *There exists a further eigenvalue* $\lambda_1 \in \left(-\frac{p_0}{k}, \frac{p_0}{k}\right)$ *with* $\lambda_1 \in \varphi$

(iii) $\inf\left\{\left|\lambda \pm \frac{p_0}{k}\right| \,\middle|\, (\lambda, u) \in \varphi\right\} = 0$

Proof: We establish the hypothesis of Theorem 2. Set

$$X: = \{u \mid u \in C^2([0, 1], \mathbf{R}),\ u(0) = u(1),\ u'(0) = u'(1)\},\quad Y: = C^\circ([0, 1], \mathbf{R}\,\Gamma),$$

and assume, that $\| \ \|_2$ respectively $\| \ \|_\infty$ are the norms on X respectively Y. Let $L: X \to Y$ be defined by: $Lu: = (pu')'$. We claim, that L is a Fredholm operator with index 0. Obviously L is continuous and $\operatorname{Ker}(L) = \{u \mid u \in X, u$ is constant$\}$ is one dimensional. Let $v \in Y$ and $\int_0^1 v(x)\,dx = 0$, then we set

$$u(x): = \int_0^x \left(\frac{1}{p(y)} \int_0^y v(z)\,dz\right) dy - x \int_0^1 \left(\frac{1}{p(y)} \int_0^y v(z)\,dz\right) dy.$$

Obviously $u \in X$ and $Lu = v$. On the other hand, if $v = Lu\,(u \in X)$, then $u'(0) = u'(1)$ implies: $\int_0^1 v(x)\,dx = 0$. Hence $R(L) = \left\{v \mid v \in Y, \int_0^1 v(x)\ dx = 0\right\}$ and therefore $\dim(Y/R(L)) = 1$. So L is a Fredholm operator with index 0. Further, following the arguments in the proof to „Satz 8" in [7], we obtain: $1(L) \geqslant p_0$. Define $A: X \to Y$ by: $Au(x): = \langle a, u \circ \sigma(x)\rangle$. A is completely continuous (Arzela-Ascoli). Since $\sum_{j=1}^m a_j \neq 0$ for $u \in \operatorname{Ker}(L), u \neq 0$. $B: X \to Y$ is defined by:

$$Bu(x): = f(x, u \circ \sigma(x), u' \circ \sigma(x), u'' \circ \sigma(x)),\ x \in [0, 1].$$

Since f is uniformly continuous on bounded sets of $\mathbf{R} \times \mathbf{R}^{3m}$ and σ is continuous, B is a continuous operator. In regard to assumption (1) we deduce analoguosly to Theorem 3 in [6], that B is a k-set-contraction. From assumption (2) we receive:

$$\lim_{\|u\|_2 \to 0} \frac{\|Bu\|_\infty}{\|u\|_2} = \lim_{\|u\|_2 \to 0} \sup_{x \in [0, 1]} \frac{|f(x, u \circ \sigma(x), u' \circ \sigma(x), u'' \circ \sigma(x))|}{\|u\|_2} = 0$$

Therefore $N: = A + B$ is a Fréchet-differentiable map in 0 with derivate A.

For applying Theorem 2 we have to show, that λ_0 is a characteristic value of (L, A) with odd multiplicity. We set $P: X \to X$ by $Pu(x): = u(0)$ and $Q: Y \to Y$ by $Qv(x): = \int_0^1 v(x)\,dx$. P is a continuous projector on $\operatorname{Ker}(L)$ and $L_P^{-1}: \operatorname{Ker}(Q) \to K(P)$ is given by

$$L_P^{-1} v(x) = \int_0^x \left(\frac{1}{p(y)} \int_0^y v(z)\,dz\right) dy - x \int_0^1 \left(\frac{1}{p(y)} \int_0^y v(z)\,dz\right) dy$$

Q satisfies the conditions $\operatorname{Ker}(Q)=R(L)$ and $R(Q)=A(\operatorname{Ker}(L))$ and is therefore the projector Q_A of Section 4. We determine $K_A=L_P^{-1}\circ(I-Q)\circ A$: For $u\in X$ we have, after changing the order of integration:

$$K_A u(x)=\int_0^x\left(\int_z^x\frac{1}{p(y)}\,dy\right)\langle a,u(z)\rangle\,dz-$$

$$-x\int_0^1\left(\int_z^1\frac{1}{p(y)}\,dy\right)\langle a,u\circ\sigma(z)\rangle\,dz-$$

$$-\left[\int_0^1\langle a,u\circ\sigma(z)\rangle\,dz\right]\left[\int_0^x\frac{y}{p(y)}\,dy\right]+$$

$$+x\left[\int_0^1\frac{y}{p(y)}\,dy\right]\left[\int_0^1\langle a,u\circ\sigma(z)\rangle\,dz\right]$$

$$=\int_0^x h_1(x,z)\langle a,u\circ\sigma(z)\rangle\,dz+\int_0^1 h_2(x,z)\langle a,u\circ\sigma(z)\rangle\,dz$$

This implies, that the eigenvalue problem $u=\lambda K_A u$ is equivalent to problem (I). Hence assumption (3) ensures, that λ_0 is a characteristic value of (I, K_A) of odd multiplicity, thus for (L, A). Therefore all hypotheses of Theorem 2 are satisfied.

Remarks:

(1) Problem (I) leads to a linear operator equation of the form $I=\lambda(V+C)$, where C is an operator with finite dimensional range. Corresponding to the case of a Volterra equation, $I-\lambda V$ is injective and therefore a linear isomorphism. Hence we can reduce (I) to an operator equation of the form $x=\lambda T(\lambda)x$, where $T(\lambda)=(I-\lambda V)^{-1}{}^\circ C$ is an operator with finite dimensional range.

(2) A second order functional differential equation of neutral type without parameter λ and with other boundary conditions is considered in [8]. In the here considered case $\sigma_j([0,1])\subseteq[0,1]$ for $2\leq j\leq m$ can be assumed without loss of generality.

(3) For convenience we have supposed that a is constant, and that the linear part is independent of u''. Equations without these restrictions can be treated analogously.

(4) For sufficient large a and suitable $\sigma_j\,(2\leq j\leq m)$ there exists an eigenvalue $\lambda_0\in\left(-\frac{p_0}{k},\frac{p_0}{k}\right)$.

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