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A THEOREM ON FIXED POINT IN LOCALLY CONVEX SPACES

Olga Hadžić, Đura Paunić

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In the papers [1] and [2] theorems on the fixed point of the mapping $T: M \to M$ are proved where M is closed and convex subset of a sequentially complete locally convex space E in which a topology is defined by a saturated family of seminorms $|\cdot|_\alpha$, $\alpha \in \mathcal{J}$ and the following condition is satisfied:

$$|Tx - Ty|_\alpha \leqslant q(\alpha)\, |x - y|_{\varphi(\alpha)} \tag{1}$$

for every $x, y \in M$ and every α where $q(\alpha) \geqslant 0$ and $\varphi: \mathcal{J} \to \mathcal{J}$.

The aim of this paper is to generalize these results when the mapping $T: M \to M$ satisfies the condition:

$$|Tx - Ty|_\alpha \leqslant \sum_{i=1}^{k} q(\alpha, i)\, |x - y|_{\varphi_i(\alpha)} \qquad \text{for every } x, y \in M \tag{2}$$

where $q(\alpha, i) \geqslant 0$ for every $(\alpha, i) \in \mathcal{J} x \{1, 2, \ldots, k\}$ and $\varphi_i: \mathcal{J} \to \mathcal{J}$ for every $i = 1, 2, \ldots, k$.

If, for instance, the mapping T is the sum of mappings T_i $(i = 1, 2, \ldots, k)$ where each T_i satisfies a condition of the type (1) then T will satisfy an inequality of the type (2).

We shall use the following symbols: $V(n, k)$ is the set of all variations with repetitions of numbers $1, 2, \ldots, k$ of the class n;

$$i_1 i_2 \ldots i_n \in V(n, k); \quad P(\alpha, 0, x) = |Tx - x|_\alpha$$

$$P(\alpha, n, x) = \max_{i_1 i_2 \ldots i_n \in V(n, k)} \{|Tx - x|_{\varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_n}(\alpha)}\} \qquad n = 1, 2, \ldots$$

$$Q(\alpha, n) = \max_{i i_1 i_2 \ldots i_n \in V(n+1, k)} \{q(\varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_n}(\alpha), i)\} \quad n = 1, 2, \ldots;$$

$$Q(\alpha, 0) = \max_{i = 1, 2, \ldots, k} q(\alpha, i)$$

$$S(\alpha, x) = P(\alpha, 0, x) + \sum_{n=2}^{\infty} k^{n-1} P(\alpha, n-1, x) \prod_{i=0}^{n-2} Q(\alpha, i)$$

and $S_m(\alpha, x)$ is the sum of the series $S(\alpha, x)$ to the m-th member (m-th partial sum).

Theorem 1. *Suppose that the following conditions are satisfied:*

1. *For every* $(\alpha, i) \in \mathcal{J} x \{1, 2, \ldots, k\}$ *there exist* $q(\alpha, i) \geqslant 0$ *and mappings* $\varphi_i : \mathcal{J} \to \mathcal{J}$ *so that the following inequality holds:*

$$|Tx - Ty|_\alpha \leqslant \sum_{i=1}^{k} q(\alpha, i) \, |x - y|_{\varphi_i(\alpha)} \qquad \text{for every } x, y \in M.$$

2. *There is* $x_0 \in M$ *such that:*

$$R = \sup_{\alpha \in \mathcal{J}} \overline{\lim_{n \in N}} \sqrt{P(\alpha, n, x_0) \prod_{i=0}^{n-1} Q(\alpha, i)} < \frac{1}{k}$$

Then there exist at least one solution x^* *of the equation* $Tx = x$. *Also the following conditions are satisfied:*

$$\lim_{n \to \infty} k^n \max_{i_1 i_2 \ldots i_n \in V(n, k)} \{ |x^* - x_0|_{\varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_n}(\alpha)} \} \prod_{i=0}^{n-1} Q(\alpha, i) = 0 \tag{3}$$

for every $\alpha \in \mathcal{J}$

$$|x^* - T^m x_0|_\alpha \leqslant S(\alpha, x_0) - S_m(\alpha, x_0) \quad \text{for every} \quad m = 1, 2, \ldots; \ \alpha \in \mathcal{J}. \tag{4}$$

Every other solution of the equation $Tx = x$ *which satisfies the condition* (3) *is identical to the solution* $x^* = \lim_{m \to \infty} T^m x_0$.

Proof: First we will show that for every $n \in N$ and every $x, y \in M$ the following inequality holds:

$$|T^n x - T^n y|_\alpha \leqslant \sum_{i_1 i_2 \ldots i_n \in V(n, k)} q(\alpha, i_n) \, q(\varphi_{i_n}(\alpha), i_{n-1}) \, q(\varphi_{i_{n-1}} \varphi_{i_n}(\alpha), \tag{5}$$

$$, i_{n-2}) \ldots q(\varphi_{i_2} \varphi_{i_3} \ldots \varphi_{i_n}(\alpha), i_1) \, |x - y|_{\varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_n}(\alpha)}$$

For $n = 1$ we have:

$$|Tx - Ty|_\alpha \leqslant \sum_{i=1}^{k} q(\alpha, i) \, |x - y|_{\varphi_i(\alpha)} = \sum_{i_1 \in V(1, k)} q(\alpha, i_1) \, |x - y|_{\varphi_{i_1}(\alpha)}$$

and so the inequality (5) holds for $n = 1$.

Suppose that the inequality (5) holds for $n - 1$ and let us prove it for n. We have:

$$|T^n x - T^n y|_\alpha = |T^{n-1}(Tx) - T^{n-1}(Ty)|_\alpha \leqslant \sum_{j_1 j_2 \ldots j_{n-1} \in V(n-1, k)} q(\alpha, j_{n-1}) \, q(\varphi_{j_{n-1}}(\alpha),$$

$$j_{n-2}) \ldots q(\varphi_{j_2} \varphi_{j_3} \ldots \varphi_{j_{n-1}}(\alpha), j_1) \, |Tx - Ty|_{\varphi_{j_1} \varphi_{j_2} \ldots \varphi_{j_{n-1}}(\alpha)}$$

Because of:

$$|Tx - Ty|_{\varphi_{j_1} \cdot \varphi_{j_2} \ldots \varphi_{j_{n-1}}(\alpha)} \leqslant \sum_{i \in V(1, k)} q(\varphi_{j_1} \varphi_{j_2} \ldots \varphi_{j_{n-1}}(\alpha), i) \, |x - y|_{\varphi_i \cdot \varphi_{j_1} \cdot \varphi_{j_2} \ldots \varphi_{j_{n-1}}(\alpha)}$$

it follows:

$$|T^n x - T^n y|_\alpha \leqslant \sum_{j_1 j_2 \ldots j_{n-1} \in V(n-1,k)} q(\alpha, j_{n-1})\, q(\varphi_{j_{n-1}}(\alpha), j_{n-2}) \ldots q(\varphi_{j_2}\varphi_{j_3}\ldots\varphi_{j_{n-1}}(\alpha),$$

$$, j_1)\Big(\sum_{i\in V(1,k)} q(\varphi_{j_1}\varphi_{j_2}\ldots\varphi_{j_{n-1}}(\alpha), i)\, |x-y|_{\varphi_i\varphi_{j_1}\ldots\varphi_{j_{n-1}}(\alpha)}\Big) =$$

$$= \sum_{j_1 j_2 \ldots j_{n-1} \in V(n-1,k)} \Big\{\sum_{i\in V(1,k)} q(\alpha, j_{n-1})\, q(\varphi_{j_{n-1}}(\alpha), j_{n-2}) \ldots q(\varphi_{j_1}\varphi_{j_2}\ldots$$

$$\ldots \varphi_{j_{n-1}}(\alpha), i)\, |x-y|_{\varphi_i\varphi_{j_1}\varphi_{j_2}\ldots\varphi_{j_{n-1}}(\alpha)}\Big\} = \sum_{i_1 i_2 \ldots i_n \in V(n,k)} q(\alpha, i_n)\, q(\varphi_{i_n}(\alpha),$$

$$, i_{n-1}) \ldots q(\varphi_{i_2}\varphi_{i_3}\ldots\varphi_{i_n}(\alpha), i_1)\, |x-y|_{\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_n}(\alpha)}$$

and the inquality (5) is proved.

Let us denote by $\Delta_n = x_n - x_{n-1}$, $n = 1, 2, \ldots$ where $x_n = Tx_{n-1}$ $(n = 1, 2, \ldots)$. Then $\Delta_{n+1} = T^n x_1 - T^n x_0$ and when the inequality (5) is applied one gets:

$$|\Delta_{n+1}|_\alpha \leqslant \sum_{i_1 i_2 \ldots i_n \in V(n,k)} q(\alpha, i_n)\, q(\varphi_{i_n}(\alpha), i_{n-1}) \ldots q(\varphi_{i_2}\varphi_{i_3}\ldots\varphi_{i_n}(\alpha), i_1) \times$$

$$\times |x_1 - x_0|_{\varphi_{i_1}\cdot\varphi_{i_2}\ldots\varphi_{i_n}(\alpha)} \leqslant \sum_{i_1 i_2 \ldots i_n \in V(n,k)} P(\alpha, n, x_0) \prod_{i=0}^{n} Q(\alpha, i-1) \leqslant$$

$$\leqslant k^n P(\alpha, n, x_0) \prod_{i=1}^{n-1} Q(\alpha, i).$$

From $R < \dfrac{1}{k}$ it follows that the series $\sum_{i=1}^{\infty} \Delta_i$ is convergent and because of $x_n = \sum_{i=1}^{n} \Delta_i + x_0$ there exists $\lim_{n\to\infty} x_n = x^*$. Using that $x_n = Tx_{n-1}$ and that T is a continuous mapping one gets that $x^* = Tx^*$ immediately.

Let us prove the inequality (3). From the inequality (5) it follows:

$$|x_m - x_0|_{\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_n}(\alpha)} \leqslant |x_1 - x_0|_{\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_n}(\alpha)} +$$

$$+ \sum_{s=2}^{m} |\Delta_s|_{\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_n}(\alpha)} \leqslant |x_1 - x_0|_{\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_n}(\alpha)} +$$

$$+ \sum_{s=2}^{m} \Big(\sum_{j_1 j_2 \ldots j_{s-1} \in V(s-1,k)} q(\varphi_{i_1}\varphi_{i_2}\varphi_{i_n}(\alpha), j_{s-1})\, q(\varphi_{j_{s-1}}\varphi_{i_1}\varphi_{i_2}\ldots$$

$$\ldots \varphi_{i_n}(\alpha), j_{s-2}) \ldots q(\varphi_{j_2}\varphi_{j_3}\ldots\varphi_{j_{s-1}}\varphi_{i_1}\varphi_{i_2}\ldots$$

$$\ldots \varphi_{i_n}(\alpha), j_1)\, |x_1 - x_0|_{\varphi_{j_1}\varphi_{j_2}\varphi_{j_{s-1}}\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_n}(\alpha)}\Big)$$

$$\leqslant P(\alpha, n, x_0) + \sum_{s=2}^{\infty} k^{s-1} P(\alpha, n+s-1, x_0) \prod_{i=0}^{s-2} Q(\alpha, n+i) =$$

$$= \frac{1}{k^n \cdot \prod_{i=0}^{n-1} Q(\alpha, i)} \Big(P(\alpha, n, x_0) k^n \prod_{i=0}^{n-1} Q(\alpha, i) + \sum_{s=2}^{\infty} k^{n+s-1} P(\alpha, n+s-1, x_0) \times$$

$$\times \prod_{i=0}^{n-1} Q(\alpha, i) \prod_{i=n}^{n+s-2} Q(\alpha, i) \Big) = \frac{1}{k^n \cdot \prod_{i=0}^{n-1} Q(\alpha, i)} \Big(\sum_{s=n+1}^{\infty} k^{s-1} P(\alpha, s-1, x_0) \times$$

$$\times \prod_{i=0}^{s-2} Q(\alpha, i) \Big) = \frac{S(\alpha, x_0) - S_n(\alpha, x_0)}{k^n \cdot \prod_{i=0}^{n-1} Q(\alpha, i)}.$$

Now we have:

$$k^n \cdot \prod_{i=0}^{n-1} Q(\alpha, i) \max_{i_1 i_2 \dots i_n \in V(n, k)} \{ | x_m - x_0 |_{\varphi_{i_1} \varphi_{i_2} \dots \varphi_{i_n}(\alpha)} \} \leqslant$$

$$\leqslant S(\alpha, x_0) - S_n(\alpha, x_0)$$

and when $m \to \infty$ it follows:

$$k^n \prod_{i=0}^{n-1} Q(\alpha, i) \max_{i_1 i_2 \dots i_n \in V(n, k)} \{ | x^* - x_0 |_{\varphi_{i_1} \varphi_{i_2} \dots \varphi_{i_n}(\alpha)} \} \leqslant$$

$$\leqslant S(\alpha, x_0) - S_n(\alpha, x_0).$$

If $n \to \infty$ we obtain (3) because of $\lim_{n \to \infty} S_n(\alpha, x_0) = S(\alpha, x_0)$. For $n > m$ we have:

$$| x_n - x_m |_\alpha \leqslant \sum_{s=m+1}^{n} | \Delta_s |_\alpha \leqslant \sum_{s=m+1}^{\infty} k^{s-1} P(\alpha, s-1, x_0) \prod_{i=0}^{s-2} Q(\alpha, i) = S(\alpha, x_0) - S_m(\alpha, x_0)$$

and when $n \to \infty$ one gets (4) since the right side does not depend on n. Suppose now that $y = Ty$ and

$$\lim_{n \to \infty} k^n \prod_{i=0}^{n-1} Q(\alpha, i) \cdot \max_{i_1 i_2 \dots i_n \in V(n, k)} \{ | y - x_0 |_{\varphi_{i_2} \varphi_{i_2} \dots \varphi_{i_n}(\alpha)} \} = 0.$$

We shall show that $x^* = y$ using the inequality (5). From $x^* = Tx$ and $y = Ty$ it follows:

$$| x^* - y |_\alpha = | T^n x^* - T^n y |_\alpha \leqslant \sum_{i_1 i_2 \dots i_n V(n, k)} q(\alpha, i_n) q(\varphi_{i_n}(\alpha), i_{n-1}) \dots$$

$$\dots q(\varphi_{i_2} \varphi_{i_3} \dots \varphi_{i_n}(\alpha), i_1) | x^* - y |_{\varphi_{i_1} \varphi_{i_2} \dots \varphi_{i_n}(\alpha)} \leqslant$$

$$\leqslant \max_{i_1 i_2 \dots i_n \in V(n, k)} \{ | x^* - x_0 |_{\varphi_{i_1} \varphi_{i_2} \dots \varphi_{i_n}(\alpha)} \} k^n \prod_{i=1}^{n} Q(\alpha, i-1) +$$

$$+\max_{i_2 i_2 \ldots i_n \in V(n,k)} \{ |y-x_0|_{\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_n}(\alpha)} \} k^n \prod_{i=1}^{n} Q(\alpha, i-1)$$

and if $n \to \infty$ we have $|x^* - y|_\alpha = 0$ for every $\alpha \in \mathcal{J}$ so $x^* = y$.

Using this theorem wo are able to generalize the result of B. Stanković [3] concerning the existence of a solution of the initial-value problem:

$$\frac{dx}{dt} = f(x, t) \qquad x(t_0) = x_0 \tag{6}$$

Let $U = \{x \mid x \in E \mid x - x_0|_{\alpha_i} \leqslant b,\ i = 1, 2, \ldots, n\}\ b > 0$, and let $\Delta = [t_0 - a, t_0 + a]$ be the closed interval, f be a continuous mapping of the product $U x \Delta$ into E.

Theorem 2. *Let the following conditions be fulfilled:*

1. $\sup_{(x,t) \in Ux\Delta} |f(x, t)|_{\alpha_i} \leqslant p < \infty \quad i = 1, 2, \ldots, n.$

2. *For every* $(\alpha, i) \in \mathcal{J} x \{1, 2. \ldots, k\}$ *there are numerical functions* $\{k_{\alpha, i}(t)\}$ *intergrable over* Δ, *mappings* $\varphi_i : \mathcal{J} \to \mathcal{J}$ *and* $h' > 0$ *such that:*

a) $|f(x, t) - f(y, t)|_\alpha \leqslant \sum_{i=1}^{k} k_{\alpha, i}(t) \, |x - y|_{\varphi_i}(\alpha)$

for every $(x, y, t) \in U^2 x \Delta$.

b) $R = \sup_{\alpha \in \mathcal{J}} \overline{\lim_{n \in N}} \sqrt[n]{P(\alpha, n) \prod_{i=1}^{n-1} Q(\alpha, i)} < \frac{1}{k}$

where:

$$P(\alpha, n) = \max_{i_1 \ldots i_n \in V(n,k)} \sup \{ |f(x_0, t)|_{\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_n}(\alpha)} \mid t \in \Delta \}$$

$$q(\alpha, i) = \max \left\{ \int_{t_0}^{t_0+h'} k_{\alpha, i}(u)\, du, \int_{t_0-h'}^{t_0} k_{\alpha, i}(u)\, du \right\}$$

$$Q(\alpha, n) = \max_{ii_1 i_2 \ldots i_n \in V(n+1, k)} \{ q(\varphi_{i_1} \ldots \varphi_{i_n}(\alpha) i) \}.$$

Then there exists a solution of the differential equation (6) *which is defined over* $[t_0 - h, t_0 + h] = h] = \Delta'$ *where* $h = \min \left(a, \frac{b}{p}, h' \right) / 2$.

Proof: We shall apply Theorem 1 taking for the space E the space $\mathcal{C}(\Delta', E)$ of all continuous mappings of the interval Δ' into the space E where topology on $\mathcal{C}(\Delta', E)$ is given by $|\tilde{x}|_\alpha = \sup_{t \in \Delta'} |x(t)|_\alpha$ and $M = \mathcal{C}(\Delta', U)$. Mapping T is given by:

$$(T\tilde{x})(t) = x_0 + \int_{t_0}^{t} f(x(u), u)\, du$$

Then following inequality holds:

$$|\tilde{Tx}-\tilde{Ty}|_\alpha \leqslant \sum_{i=1}^{k} \sup_{t\in\Delta}\left\{\left|\int_{t_0}^{t} k_{\alpha,i}(u)\,|x(u)-y(u)|_{\varphi_i(\alpha)}du\right|\right\}\leqslant$$

$$\leqslant \sum_{i=1}^{k}\max\left\{\int_{t_0}^{t_0+h'} k_{\alpha,i}(u)\,du,\ \int_{t_0-h'}^{t_0} k_{\alpha,i}(u)\,du\right\}|\tilde{x}-\tilde{y}|_{\varphi_i(\alpha)}=$$

$$=\sum_{i=1}^{k} q(\alpha,i)\,|\tilde{x}-\tilde{y}|_{\varphi_i(\alpha)}$$

and since $h'<\frac{b}{p}$, $T\tilde{x}\in\mathscr{C}(\Delta',U)$ for every $\tilde{x}\in\mathscr{C}(\Delta',U)$. Reimaining conditions of the Theorem 1 can be verified easily. Applying it one gets that there is a solution of the initial-value problem (6).

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